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CONCEPT OF THE MONTH
Nature of Roots

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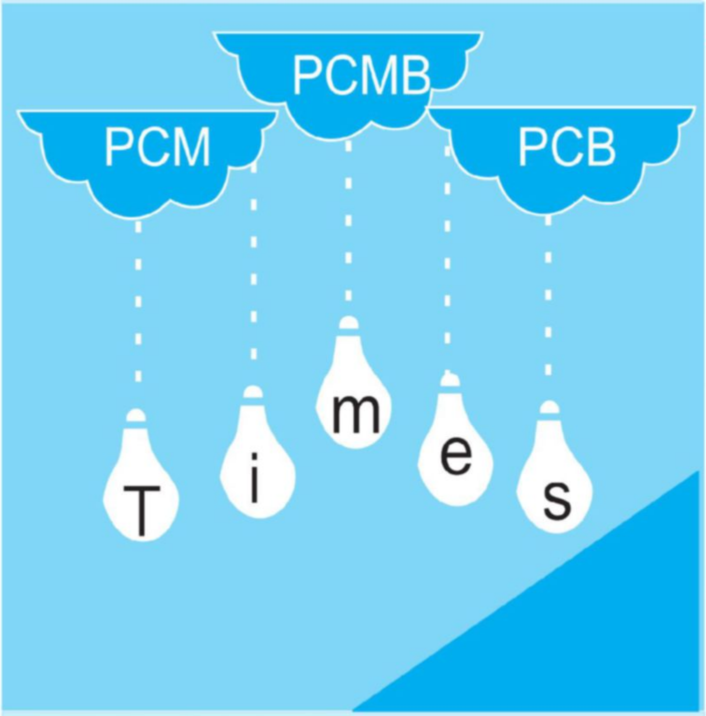

CONTENTS

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Nature of Roots

Concept of the month

This column is aimed at preparing students for all competitive exams like JEE, BITSAT etc. Every concept has been designed by highly qualified faculty to cater to the needs of the students by discussing the most complicated and confusing concepts in Mathematics.

By. DHANANJAYA REDDY THANAKANTI
(Bangalore)

Introduction:

The nature of roots is a simple concept,which categorises the roots of a given polynomial into imaginary, real, unequal or equal with out finding actual roots of the Polynomial.

Quadratic Equation: In quadratic equation , the nature of roots can be obtained from the discriminate of the quadratic equation. The discriminant tells us what kind of solutions to expect when solving quadratic equation.

It is possible to predict whether or not the roots of a quadratic equation are real or unreal, or if there exists double roots without explicitly solving for the roots.

The discriminant of a quadratic equation of the form $ax^2+bx+c=0$ is the quantity b^2-4ac. When a, b, c are real, this is a notable quantity, because if the discriminant is positive, the equation has two real roots; if the discriminant is negative, the equation has two nonreal roots; and if the discriminant is 0, the equation has a real double root.

Polynomials of degree n

The discriminant can tell us something about the roots of a given polynomial $p(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_0$ of degree n with all the coefficients being real. But for polynomials of degree 4 or higher it can be difficult to use it.So using following theorems we can predict the nature of roots.

Descartes Rule of Signs: Descarte's rule of signs is a method used to determine the number of positive and negative roots of a polynomial. The rule gives an upper bound on the number of positive or negative roots, but does not specify the exact amount.

Positive Roots: If the terms of a polynomial with real coefficients are ordered by descending variable exponent, then the number of positive roots of the polynomial is equal to the number of sign differences between consecutive nonzero coefficients or is less than that by an even number. Multiple roots are counted separately.

Negative Roots: The bound for negative roots is a corollary of the positive root bound. The number of negative roots is the number of sign changes after multiplying the coefficients of odd-power terms by -1, or fewer than that by a positive even number.

Rational Root Theorem: Given a polynomial $p(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0$ with integral coefficients, $a_n\neq 0$. The Rational Root Theorem states that if $P(x)$ has a rational root $r = \pm \frac{p}{q}$ with *p, q* relatively prime positive integers, *p* is a divisor of a_0 and *q* is a divisor of a_n.

As a consequence, every rational root of a monic polynomial with integral coefficients must be integral.

Multiple Root: A multiple root is a root with multiplicity $n \geq 2$, also called a multiple point or repeated root. If a polynomial has a multiple root, its derivative also shares that root.

Location of Roots Theorem: The location of roots theorem is one of the most intutively obvious properties of continuous functions, as it states that if a continuous function attains positive and negative values, it must have a root (i.e. it must pass through 0).

Applications of the Intermediate Value Theorem: We can use the Intermediate Value Theorem (IVT) to show that certain equations have solutions, or that certain polynomials have roots.

Corollary of Rolle's Theorem: Between any two roots of a differentiable function there is at least one root of its derivative.

Exercise

1. Let $a \in R$ and let $f: R \to R$ be given by $f(x) = x^5 - 5x + a$. then
   (a) $f(x)$ has three real roots if $a > 4$
   (b) $f(x)$ has only real root if $a > 4$
   (c) $f(x)$ has three real roots if $a < -4$
   (d) $f(x)$ has three roots if $-4 < a < 4$
2. Corresponding to the equation $\left(\frac{x}{x+1}\right)^2 + \left(\frac{x}{x-1}\right)^2 = a(a-1)$ mark the wrong option
   (a) four real roots, if $a > 2$
   (b) two real roots if $1 < a < 2$
   (c) no real roots if $a < -1$
   (d) four real roots if $a < -1$
3. The total number of values of a so that $x^2 - x - a = 0$ has integral roots, where $a \in N$ and $6 \leq a \leq 100$, is equal to
   (a) 2 (b) 4 (c) 6 (d) 8
4. Prove that $3x^5 - 5x^3 + a = 0$
   i) three real roots when $-2 < a < 2$
   ii) one real root and 4 complex roots when $a > 2 \, or \, a < -2$ Can you tell the sign of the real root?
5. Prove that when $a, b > 0, n \geq 2$ and $a^n > b^{n-1}$, the equation $x^n - nax + n(n-1)b = 0$ has
   i) exactly two real roots if n is even.
   ii) exactly three real roots if n is odd
6. Prove that the equation $x^6 - 6a^5x + 5 = 0$ has
   i) exactly two real roots when $a > 1$.
   ii) Two real and equal roots when $a = 1$.
   iii) no real roots when $a < 1$.
7. Prove that when $a > 1$, the equation $x^5 - 5a^4x + 4 = 0$ increases from $-\infty \, to + \infty$.
8. Consider a Cubic equation, $x^3 - px + 2 = 0, p \in R$. Analyse the nature of roots of the equation for different real values of *p*. Then the given cubic has

(i) $p < 3$	(a) three distinct real roots
(ii) $p > 3$	(b) unique negative real root
(iii) $0 < p < 3$	(c) unique positive real root
(iv) $p < 0$	(d) a root of multiplicity 2
(v) p=1	

9. Match the following for the equation $x^2 + a|x| + 1 = 0$. where a is a parameter.

Column I	**Column II**
(a) No real roots	(p) $a < -2$
(b) Two real roots	(q) ϕ
(c) Three real roots	(r) $a = -2$
(d) Four distinct real roots	(s) $a \geq 0$

HINTS & SOLUTIONS

1.Sol: (b,d) $f(x)=x^5-5x+a$

$f(x)=0\Rightarrow x^5-5x+a=0$

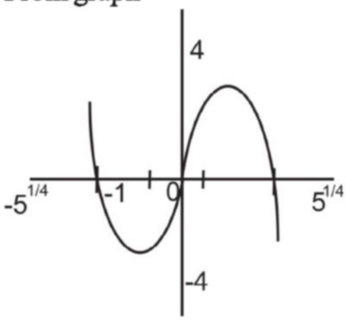
$\Rightarrow a=5x-x^5=g(x)$

$\Rightarrow g(x)=0$ when $x=0,5^{1/4},-5^{1/4}$ and

$g'(x)=0\Rightarrow x=1,-1$

Also $g(-1)=-4\, and\, g(1)=4$

$\therefore$ graph of $g(x)$ will be as shown below. From graph



if $a\in(-4,4)$

then $g(x)=a$ is

$f(x)=0$ has 3 real roots

If $a>4\, or\, a<-4$

then $f(x)=0$ has only one real root.

$\therefore$ (b) and (d) are the correct options.

2.Sol: (c)

$$\left(\frac{x}{x+1}+\frac{x}{x-1}\right)^2-2\left(\frac{x}{x+1}\right)\left(\frac{x}{x-1}\right)=a(a-1)$$

when $a<-1\Rightarrow$ All roots are real

$$\left(\frac{2x^2}{x^2-1}\right)-\frac{2x^2}{x^2-1}=a(a-1)$$

when $1<a<2\Rightarrow$ only two real roots

$z^2-z-a(a-1)=0$, where $z=\dfrac{2x^2}{x^2-1}$

when $a>2\Rightarrow$ all roots are real

$z=a\, or\, z=1-a$

$$\frac{2x^2}{x^2-1}=a\, or\, \frac{2x^2}{x^2-1}=1-a$$

$$x=\pm\sqrt{\frac{a}{a-2}}\ \text{or}\ x=\pm\sqrt{\frac{a-1}{a+1}}$$

3.Sol:(d) $x^2-x-a=0, D=1+4a=odd$

D must be perfect square of some odd integer.

Let $D=(2\lambda+1)^2$

$\Rightarrow 1+4a=1+4\lambda^2+4\lambda$

$\Rightarrow a=\lambda(\lambda+1).$

Now, $a\in[6,100]$

$\Rightarrow a=6,12,20,30,42,56,72,90$

Thus a can attain eight different values.

4.Sol: The equation is $p(x)=0$ (1)

where $p(x)=3x^5-5x^3+a$

$p'(x)=15(x^4-x^2)$

$=15x^2(x+1)(x-1)$

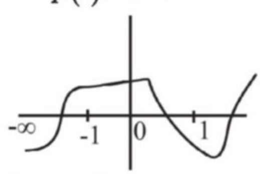
$p'(x)=0$

$\Rightarrow x=0,0,-1,1$

$p(0)=a$

$p(-1)=a+2$

$p(1)=a-2$



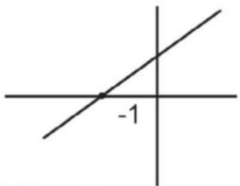
i) $-2<a<2$

x	$-\infty$	-1	0	1	∞
$sign\, of\, p(x)$	$-\infty$	$+ve$	$can't\, say$	$-ve$	$+ve$

Thus $p(x)$ canges sign in each of the intervals $(-\infty,-1),(-1,1)$ and $(1,\infty)$, so tha Eq (1) has at least one (or an odd number of) root in each of these intervals, hence at least 3 real roots. Since there are 3 turning points of $p(x)$, therefore Eq (1) cannot have more than 4 real roots.

If any of the intervals has more than one real root, it will have at least 3 real roots. So that the total number of real roots will be least 5, which is not possible.
Hence there are exactly 3 real roots one in each of the subintervals.

ii) Two cases arise:



Case 1: $a > 2$

x	$-\infty$	-1	0	$+1$	∞
sign of $p(x)$	$-ve$	$+ve$	$+ve$	$+ve$	$+ve$

There is a change of sign in the interval $(-\infty,-1)$ only so that Eq (1) has at least one root in this interval. Since there is no turning point in this interval, there is exactly one root in $(-\infty,-1)$. The root is negative.
The other 4 roots are complex.

Case 2: $a < -2$

x	$-\infty$	-1	0	$+1$	∞
sign of $p(x)$	$-ve$	$-ve$	$-ve$	$-ve$	$+ve$

There is a change of sign in the interval $(1,\infty)$ only, so that (1) has at least one root in this interval. Since there is no turning point in this interval, there is exactly one root in $(1,\infty)$. The root is positive. The other 4 roots must be complex.

5.Sol: The given equation is

$$p(x)=0 \qquad (1)$$

Where $p(x)=x^n-nax+(n-1)b$

$$p'(x)=nx^{n-1}-na$$
$$=n\left(x^{n-1}-a\right)$$

$$p'(x)=0$$
$$\Rightarrow n\left(x^{n-1}-a\right)=0$$
$$\Rightarrow x=a^{\frac{1}{n-1}}cis\left(\frac{2k\pi}{n-1}\right),k=0,1,....,n-2$$

where $\cos\theta=\cos\theta+i\sin\theta$

i) n is even,
In this case, $n-1$ is odd, so that Eq (2) has only one real root, namely

Now $$p(\alpha)=\alpha^n-na\alpha+(n-1)b$$
$$=a^{\frac{n}{n-1}}-na^{\frac{n}{n-1}}+(n-1)b$$
$$=-(n-1)\left[a^{\frac{n}{n-1}}-b\right]$$

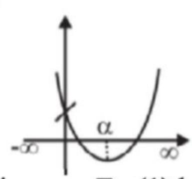
$$<0\left(\because n\geq 2,a^n>b^{n-1}\right)$$

Thus $p(\alpha)<0,p'(\alpha)=0$

x	$-\infty$	α	∞
sign of $p(x)$	$+ve$	$-ve$	$+ve$

since $p(x)$ changes sign in each of the intervals $(-\infty,\alpha)$ and (α,∞), so that there is at least one real root root in each of these in - tervals. Since there is only one turning point Eq (1) cannot have more than 2 real roots. Thus there are exactly two real roots.

ii) n is odd



$\therefore$ $n-1$ is even. Eq (1) has 2 real roots.,

$$x=\pm a^{\frac{1}{n-1}}$$
$$=\pm\alpha(say)$$

$$\therefore p(\alpha)=\alpha^n-na\alpha+(n-1)b$$

$$= -(n-1)\left[a^{\frac{n}{n-1}} - b\right] < 0$$

similarly $p(-\alpha) = (n-1)\left[a^{\frac{n}{n-1}} + b\right] > 0$

Thus $p(\alpha) < 0$, $p(-\alpha) > 0$

$p'(\alpha) = p'(-\alpha) = 0$

x	$-\infty$	$-\alpha$	α	∞
sign of $p(x)$	$-ve$	$+ve$	$-ve$	$+ve$

Since $p(x)$ changes sign in each of the intervals $(-\infty,-\alpha)$, $(-\alpha,\alpha)$ and (α,∞), Eq (1) has atleast one root in each of these intervals. Also $p(x)$ has only 2 turning points so that it cannot have more than 3 real roots.

6.Sol: The equation is

$$p(x) = 0 \qquad (1)$$

$$p(x) = x^6 - 6a^5x + 5$$

$$p'(x) = 6x^5 - 6a^5$$

$$p'(x) = 6\left(x^5 - a^5\right)$$

$p'(x) = 0$

$\Rightarrow x^5 - a^5 = 0$

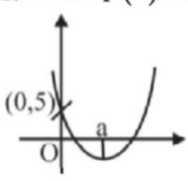
$\Rightarrow x = a\, cis\dfrac{2k\pi}{5}, k = 0,1,2,3,4.$

where cis $\theta \equiv \cos\theta + i\sin\theta$. Thus, $p'(x) = 0$ has only one real root, namely $x = a$ has at the most two real roots.

$p(a) = -5a^5 + 5$

$= 5\left(1 - a^5\right)$

i) $a > 1$. Then $p(a) < 0$



x	$-\infty$	a	∞
sign of $p(x)$	$+ve$	$-ve$	$+ve$

$p(x)$ changes sign in $(-\infty,a)$ and (a,∞) so that $p(x)$ has at least one root in each of these intervals.

Hence it has atleast two real roots. Since there is only one turning point, it cannot have more than two real roots.

$\therefore$ equation (1)has exactly 2 real roots.

ii) $a = 1$.

$p'(x) = 5x^4 - 5a^4 = 5\left(x^4 - a^4\right)$

$= 5\left(x^2 + a^2\right)(x+a)(x-a)$

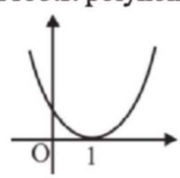
Real roots of $p'(x) = 0$ are $\pm a$ Thus $a, -a$ are the turing point of $p(x)$.

$p(-a) = 4\left(a^5 + 1\right) > 0$

$p(a) = -4\left(a^5 - 1\right) < 0$ *as* $a > 1$.

x	$-\infty$	$-a$	a	∞
sign of $p(x)$	$-ve$	$+ve$	$-ve$	$+v$

Thus $p(x)$ has change of sign in each of the intervals $I_1 = (-\infty,-a), I_2 = (-a,a), I_3 = (a,\infty)$ so that it has a root in each of these intervals. Thus $p(x)$ has at least 3 real roots. Between any two real roots of $p(x) = 0$ there is a real root of $p'(x) = 0$ and $p'(x) = 0$ has exactly two real roots $\therefore p(x) = 0$ can not have more than 3 real roots. polynomial of degree ≥ 1.



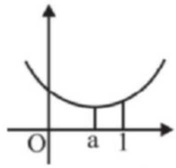
Then $p'(1) = 0$, $p(1) = 0$

Also 1 is a simple (not repeated) root of $p'(x) = 0$.

$\therefore 1$ is a double root of Eq (1)

Thus Eq (1) has two real equal roots, namely x=1

iii) $a<1$.



In this case $p(a)>0$.

x	$-\infty$	a	∞
$sign\,of\ p(x)$	$+ve$	$+ve$	$+ve$

since there is no change of signs, (1) has no real root.

7.Sol: $p'(x)=5x^4-5a^4$

$=5(x^4-a^4)$

$=5(x^2+a^2)(x+a)(x-a)$

Real roots of $p'(x)=0$ are $\pm a$ Thus $a,-a$ ar the turing point of $p(x)$.

$p(-a)=4(a^5+1)>0$

$p(a)=-4(a^5-1)<0\ as\ a>1.$

x	$-\infty$	$-a$	a	∞
$sign\,of\ p(x)$	$-ve$	$+ve$	$-ve$	$+ve$

Thus $p(x)$ has change of siign in each of the intervals

$$I_1=(-\infty,-a),I_2=(-a,a),I_3=(a,\infty)$$

so that it has a root in each of these intervals. Thus $p(x)$ has at least 3 real roots.

Between any two real roots of $p(x)=0$ there is a real root of $p'(x)=0$ and $p'(x)=0$ has exactly two real roots $\therefore p(x)=0$ can not have more than 3 real roots.

8. Sol:

(i) $\rightarrow$ b
(ii) $\rightarrow$ a
(iii) $\rightarrow$ b
(iv) $\rightarrow$ b
(v) $\rightarrow$ d

The given equaiton is $x^3-px+2=0$, which can be rewritten as $y=p=x^2+\frac{2}{x},p\in R$

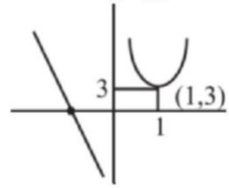
$y=p,y\in R$ is a set of lines parallel to x-axis

The graph of $y=x^2+\frac{2}{x}$:

Observations:

i) As $x\to\pm\infty,y\to x^2$. As x becomes larger, the sketech behaves like that of a parabola i.e, the first factor x^2 dominates the second factor $\frac{2}{x}$.

ii) As $x\to 0+,y\to\infty$ and as $x\to 0-,y\to-\infty$ for $y=x^2+\frac{2}{x},\frac{dy}{dx}=2x-\frac{2}{x^2}$



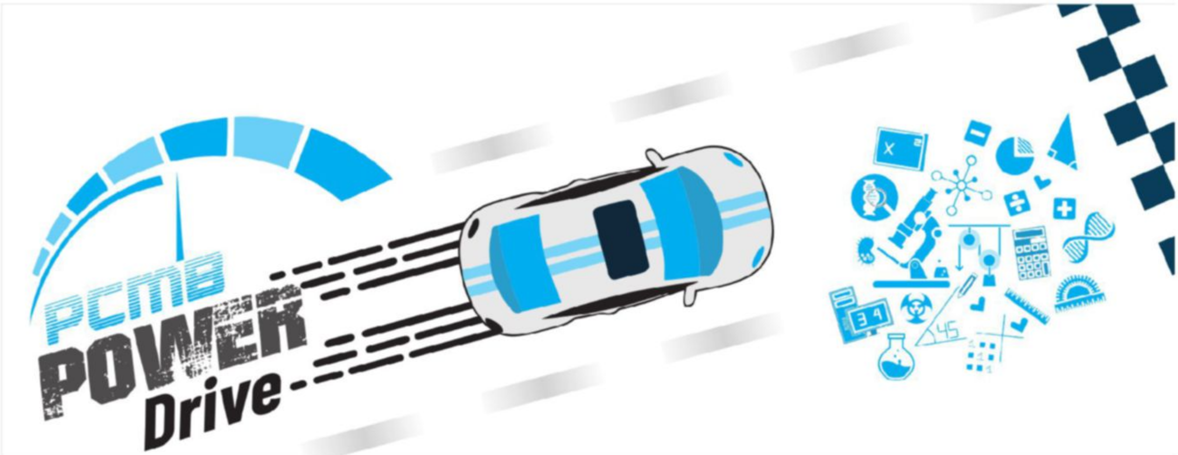
At $(1,3)$'y' has a turning point as $\frac{dy}{dx}$ vanishes at $(1,3)$. The graph of $y=x^2+\frac{2}{x}$, is as shown above.

9.Sol: $a\to s;b\to r;c\to q;d\to p$ when $a\ge 0$, we have no roots as all the terms are followed by +ve sign. Also for a = -2, we have

$x^2-2|x|+1=0$ or $|x|-1=0\Rightarrow x=\pm 1\ a<-2$, for given equation

$$|x|=\frac{-a\pm\sqrt{a^2-4}}{2}>0$$

$|-a|>\sqrt{a^2-4}$. Obviously, the equation has no three real roots for any value of a.



STRAIGHT LINES

Distance: Distance between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Section Formula: The coordinates of a point dividing the line segment joining the points (x_1, y_1) and (x_2, y_2) internally,in the ratio m: n are $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$.

Mid Point: In particular, if $m = n$, the coordinates of the mid-point of the line segment joining the points (x_1, y_1) and (x_2, y_2) are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Area: Area of the triangle whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is

$$\frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

Slope of a Line:

Definition. If θ is the inclination of a line l, then $\tan\theta$ is called the slope or gradient of the line l.

The slope of a line whose inclination is $90°$ is not defined.

The slope of a line is denoted by m.Thus, $m = \tan\theta, \theta \neq 90°$.

(1) Two points are given: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on non-vertical line l whose inclination is θ. Obviously, $x_1 \neq x_2$,

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Parallelism and Perpendicularity : In a coordinate plane, suppose that non-vertical lines l_1 and l_2 have slopes m_1 and m_2, respectively.

(1) Hence, two non vertical lines l_1 and l_2 are parallel if and only if their slopes are equal.

(2) Two non-vertical lines are perpendicular to each other if and only if their slopes are negative reciprocals of each other,

i.e., $m_1 m_2 = 1$

Angle between Two Lines: Thus, the acute angle (say θ) between lines L_1 and L_2 with slopes m_1 and m_2, respectively, is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|, \text{ as } 1 + m_1 m_2 \neq 0$$

The obtuse angle (say ψ) can be found by using $\psi = 180° - \theta$.

Collinearity of three points: If A, B and C are three points in the XY- plane, then they will lie on a line, i.e., three points are collinear if and only if slope of AB = slope of BC.

Various Forms of the Equation of a Line:

Horizontal and vertical lines : If a horizontal line L is at a distance a from the x-axis then ordinate of every point lying on the line is either a or $-a$. Therefore, equation of the line L is either $y=a$ or $y=-a$. Choice of sign will defined upon the position of the line according as the line is above below the y-axis. Similarly, the equation of a vertical line at a distance b from the y-axis is either $x=b$ or $x=-b$.

Point-slope form: The point (x, y) lies on the line with slope m through the fixed point (x_0, y_0), if and only if, its coordinates satisfy the equation

$$y-y_0=m(x-x_0)$$

Two-Point form: The equation of the line passing through the points (x_1,y_1) and (x_2,y_2) is given by

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

Slope-Intercept form:

(1) The point (x, y) on the line with slope m and y"intercept c lies on the line if and only if $y=mx+c$

(2) Suppose L with slope m makes x - intercept d. Then equation L is

$$y=m(x-d)$$

Intercept - form: Thus, equation of the line making intercepts a and b on x- and y- axis, respectively, is

$$\frac{x}{a}+\frac{y}{b}=1$$

Normal form: The equation of the line having normal distance p from the origin and angle ω which the normal makes with the positive direction of x-axis is given by

$$x\cos\omega+y\sin\omega=p$$

General Equation of a Line: Any equation of the form $Ax+By+C=0$, where A and B are not zero simultaneously is called general linear equation or general equation of a line.

Different forms of $Ax+By+C=0$: The general equation of a line can be reduced into various forms of the equation of a line, by the following procedures:

Slope-Intercept form: If $B\neq 0$, then $Ax+By+C=0$ can be written as $y=-\frac{A}{B}x-\frac{C}{B}$

where $m=-\frac{A}{B}$ and $c=-\frac{C}{B}$.

Intercept form: If $C\neq 0$, then $Ax+By+C=0$ can be written as

$$\frac{x}{-\frac{C}{A}}+\frac{y}{-\frac{C}{B}}=1$$

where $a=-\frac{C}{A}$ and $b=-\frac{C}{B}$.

Normal form: The normal form of the equation $Ax+By+C=0$ is $x\cos\omega+y\sin\omega=p$ where

$\omega=\pm\frac{A}{\sqrt{A^2+B^2}}, \sin\omega=\pm\frac{B}{\sqrt{A^2+B^2}}$, and

$p=\pm\frac{C}{\sqrt{A^2+B^2}}$ Proper choice of signs is made so that p should be positive.

Distance of a Point From a Line: The perpendicular distance (d) of a line $AX+BY+C=0$ from a point $)\ x_1,y_1)$ is given by

$$d=\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}$$

Distance between Two Parallel Lines: The distance d between two parallel lines $y=mx+c_1$ and $y=mx+c_2$ is given by

$$d=\frac{|c_1-c_2|}{\sqrt{1+m^2}}$$

If lines are given in general form, i.e., $Ax+By+C_1=0$ and $Ax+By+C_2=0$, then above formula will take the form

$$d=\frac{|C_1-C_2|}{\sqrt{A^2+B^2}}$$

Exercise

1. The point on the lines $3x+4y=5$ which is equidistant from (1, 2) and $(3,4)$ is

(1) $(7,-4)$ (2) $(15,-10)$

(3) $(1/7,8/7)$ (4) $(0,5/4)$

2. If $2a+b+3c=0$, then the line $ax+by+c=0$ passes through the fixed point that is

(1) $\left(\frac{2}{3},1\right)$ (2) $(0,1)$

(3) $\left(\frac{2}{3},0\right)$ (4) None of these

3. If (x,y) is equidistant from $(a+b,b-a)$ and $(a-b,a+b)$, then

(1) $ax+by=0$ (2) $ax-by=0$

(3) $bx+ay=0$ (4) $bx-ay=0$

4. The angle between lines $\sqrt{3}x+y=1$ and $x+\sqrt{3}y=1$ is

(1) $\frac{\pi}{6}$ (2) $\frac{3\pi}{4}$ (3) $\frac{5\pi}{2}$ (4) $\frac{\pi}{3}$

5. Joint equation of pair of line through $(3,-2)$ and parallel to $x^2-4xy+3y^2=0$ is

(1) $x^2+3y^2-4xy-14x+24y+45=0$

(2) $x^2+3y^2+4xy-14x+24y+45=0$

(3) $x^2+3y^2+4xy-14x+24y-45=0$

(4) $x^2+3y^2+4xy-14x-24y-45=0$

6. A line passing through origin and is perpendicular to two given lines $2x+y+6=0$ and $4x+2y-9=0$. The ratio in which the origin divides this line, is

(1) 1 : 2 (2) 2 : 1 (3) 4 : 2 (4) 4 : 3

7. If the line $\frac{x}{a}+\frac{y}{b}=1$ moves such that $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$, then the locus of the foot of the perpendicular from the origin to the line is

(1) Straight line (2) Circle

(3) Parabola (4) Ellipse

8. If $A(-1,2),B(5,1),C(6,5)$ are the vertices of a parallelogram $ABCD$. The equation to the diagonal through B is

(1) $x+y+6=0$ (2) $x+y-6=0$

(3) $x-y-4=0$ (4) $x-2y-1=0$

9. The points $(-1,0)$ and $(-2,1)$ are the two extremities of a diagonal of a parallelogram. If $(-6,5)$ is the third vertex, then the fourth vertex of the parallelogram is

(1) $(2,-6)$ (2) $(2,-5)$

(3) $(3,-4)$ (4) $(-3,4)$

10. The area (in square unit) of the triangle formed by $x+y+1=0$ and the pair of straight lines $x^2-3xy+2y^2=0$ is

(1) $\frac{7}{12}$ (2) $\frac{5}{12}$ (3) $\frac{1}{12}$ (4) $\frac{1}{6}$

11. If $a>0,b>0$ the maximum area of the triangle formed by the points $O(0,0)$, $A(a\cos\theta,b\sin\theta)$ and $B(a\cos\theta,-b\sin\theta)$ is (in sq unit)

(1) $\frac{ab}{2}$ when $\theta=\frac{\pi}{4}$

(2) $\frac{3ab}{4}$ when $\theta=\frac{\pi}{4}$

(3) $\frac{ab}{2}$ when $\theta=-\frac{\pi}{2}$

(4) a^2b^2

12. The circumcentre of a triangle formed by the lines $xy+2x+2y+4=0$ and $x+y+2=0$, is

(1) $(0,-1)$ (2) $(-1,0)$

(3) $(1,1)$ (4) $(-1,-1)$

13. The two vertices of triangle are $(2,-1),(3,2)$ and the third vertex lies on $x+y=5$. The area of the triangle is 4 units, then the third vertex is

(1) (0, 5) or (1, 4) (2) (5, 0) or (4, 1)

(3) (5, 0) or (1, 4) (4) (0, 5) or (4, 1)

14. The product of the perpendicular from $(-1,2)$ to the pair of lines

$2x^2-5xy+2y^2+3x-3y+1=0$

(1) $\frac{5}{12}$ (2) $\frac{12}{5}$ (3) $\frac{6}{5}$ (4) $\frac{5}{6}$

15. If slopes of lines represented by $kx^2+5xy+y^2=0$ differ by 1, then $k=$

(1) 2 (2) 3 (3) 6 (4) 8

16. If the equation $12x^2+7xy-py^2-18x+qy+6=0$ represents a pair of perpendicular straight lines, then

(1) $p=12,q=-1$ (2) $p=-12,q=1$

(3) $p=12,q=1$ (4) $p=1,q=1$

17. The equation $12x^2+7xy+ay^2+13x-y+3+3=0$ represents a pair of perpendicular lines. Then the value of 'a' is

(1) $\frac{7}{2}$ (2) -19 (3) -12 (4) 12

18. If the angle bewteen two lines represented by $2x^2+5xy+3y^2+7y+4=0$ is $\tan^{-1}m$, then m is equal to

(1) $1/5$ (2) 1 (3) 7/5 (4) 7

19. If the equation $4x^2+hxy+y^2=0$ represent coincident lines, then h is equal to

(1) 1 (2) 3 (3) 2 (4) 4

20. Find the equation of the bisector of the obtuse angle between the lines $3x-4y+7=0$ and $-12x-5y+2=0$.

(1) $21x+77y-101=0$

(2) $99x-27y+81=0$

(3) $21x-77y+101=0$

(4) None of these

21. If the slope of one of the lines given by $ax^2+2hxy+by^2=0$ is two times the other, then

(1) $8h^2=9ab$ (2) $8h^2=9ab^2$

(3) $8h=9ab$ (4) $8h=9ab^2$

22. The equation of perpendicular bisectors of sides AB and AC of a ΔABC are $x-y+5=0$ and $x+2y=0$ respectively. If the coordinates of vertex A are $(1,-2)$, the equation of BC is

(1) $14x+23y-40=0$

(2) $14x-23y+40=0$

(3) $23x+14y-40=0$

(4) $23x-14y+40=0$

23. The angle between the lines $x^2-xy-6y^2-7x+31y-18=0$ is

(1) $\frac{\pi}{4}$ (2) $\frac{\pi}{6}$ (3) $\frac{\pi}{2}$ (4) $\frac{\pi}{3}$

24. One possible condition for the three points $(a,b)(b,a)$ and $(a^2,-b^2)$ to be collinear, is

(1) $a-b=2$ (2) $a+b=2$

(3) $a=1+b$ (4) $a=1-b$

25. The angle between the lines represented by the equation $2x^2+3xy-5y^2=0$, is

(1) $\frac{\pi}{3}$ (2) $\frac{\pi}{2}$

(3) $\tan^{-1}\left(\frac{12}{5}\right)$ (4) $\tan^{-1}\left(-\frac{7}{3}\right)$

26. If θ is the angle between the pair of straight lines $x^2-5xy+4y^2+3x-4=0$, then $\tan^2\theta$ is equal to

(1) $\frac{9}{16}$ (2) $\frac{16}{25}$ (3) $\frac{9}{25}$ (4) $\frac{21}{25}$

27. The number of rational values of m for which the y-coordinate of the point of intersection of the lines $3x+2y=10$ and $x=my+2$ is an integer is

(1) 2 (2) 4 (3) 6 (4) 8

28. Area of the traingle with vertices $(-2,2)$, $(1,5)$ and $(6,-1)$ is

(1) 15 (2) 3/5 (3) 29/2 (4) 33/2

29. If the pair of lines $x^2-2nxy-y^2=0$ and $x^2-2mxy-y^2=0$ are such that one of them represents the bisectors of the angles between the other, then

(1) $\frac{1}{n}+\frac{1}{m}=0$ (2) $\frac{1}{n}-\frac{1}{m}=0$

(3) $nm-1=0$ (4) $nm+1=0$

30. If the points (1, 0), (0, 1) and (x, 8) are collinear, then the value of x is equal to

(1) 5 (2) –6 (3) 6 (4) –7

31. The equations of the straight lines passing through the point (4, 3) and making intercepts on the coordinate axes whose sum is -1, is

(1) $\frac{x}{2}-\frac{y}{3}=1$ and $\frac{x}{-2}+\frac{y}{1}=1$

(2) $\frac{x}{2}-\frac{y}{3}=-1$ and $\frac{x}{-2}+\frac{y}{1}=-1$

(3) $\frac{x}{2}+\frac{y}{3}=1$ and $\frac{x}{2}+\frac{y}{1}=1$

(4) None of these

32. The coordinate of the point dividing internally the line joining the points $(4,-2)$ and (8, 6) in the ratio 7:5 is

(1) (16, 18) (2) (18, 16)

(3) $\left(\frac{19}{3},\frac{8}{3}\right)$ (4) $\left(\frac{8}{3},\frac{19}{3}\right)$

33. The distance of the point (1, 1) from the line $2x-3y-4=0$ in the direction of the line $x+y=1$, is

(1) $\sqrt{2}$ (2) $5\sqrt{2}$ (3) $-\frac{1}{\sqrt{2}}$ (4) $\frac{1}{2}$

34. The area of the triangle formed by the points $(a,b+c),(b,c+a),(c,a+b)$ is

(1) abc (2) $a^2+b^2+c^2$

(3) $ab+bc+ca$ (4) 0

35. If y-intercept of the line $4x-ay=8$ is thrice its x-intercept, then the value of a is equal to

(1) $\frac{3}{4}$ (2) $\frac{4}{3}$ (3) $-\frac{3}{4}$ (4) $-\frac{4}{3}$

36. A ray of light along $x+\sqrt{3}y=\sqrt{3}$ gets reflected upon reaching x-axis, the equation of the reflected ray is

(1) $\sqrt{3}y=x-\sqrt{3}$ (2) $y=\sqrt{3}x-\sqrt{3}$

(3) $\sqrt{3}y=x-1$ (4) $y=x+\sqrt{3}$

37. The equation of the straight line which passes through the intersection of the lines $x-y-1=0$ and $2x-3y+1=0$ and is parallel to x-axis, is

(1) $y=3$ (2) $y=-3$

(3) $x+y=3$ (4) None of these

38. The equation of the line passing through (a,b) and parallel to the line $\frac{x}{a}+\frac{y}{b}=1$ is

(1) $\frac{x}{a}+\frac{y}{b}=3$ (2) $\frac{x}{a}+\frac{y}{b}=2$

(3) $\frac{x}{a}+\frac{y}{b}=0$ (4) $\frac{x}{a}+\frac{y}{b}+2=0$

39. If the sum of slopes of the lines given by $x^2-4pxy+8y^2=0$ is three times their product then p has the value

(1) $\frac{1}{4}$ (2) 4 (3) 3 (4) $\frac{3}{4}$

40. The area of the triangular region whose sides are $y=2x+1, y=3x+1$ and $x=4$ is

(1) 5 (2) 6 (3) 7 (4) 8

41. The pair of lines $\sqrt{3}x^2-4xy+\sqrt{3}y^2=0$ are rotated about the origin by $\frac{\pi}{6}$ in the anticlockwise sense. The equation of the pair in the new position is

(1) $x^2-\sqrt{3}xy=0$ (2) $xy-\sqrt{3}y^2=0$

(3) $\sqrt{3}x^2-xy=0$ (4) None of these

42. Orthocentre of the triangle formed by the lines $x+y=1$ and $xy=0$ is

(1) (0, 0) (2) (0, 1)

(3) (1, 0) (4) $(-1,1)$

43. If $x^2+y^2+2gx+2fy+1=0$ represents a pair of straight lines, then f^2+g^2 is equal to

(1) 0 (2) 1 (3) 2 (4) 4

44. The reflection of the point $(-3,2)$ with respect to the line $y+5=0$ is

(1) $(-5,-2)$ (2) $(-1,-2)$

(3) $(-3,-12)$ (4) $(-3,-2)$

45. The slopes of the straight line $\frac{x}{10}-\frac{y}{4}=3$ is

(1) $\frac{5}{2}$ (2) $\frac{-5}{2}$ (3) $\frac{2}{5}$ (4) $\frac{-2}{5}$

ANSWER KEY

1. 2	2. 4	3. 4	4. 1	5. 1
6. 4	7. 2	8. 2	9. 3	10. 3
11. 1	12. 4	13. 3	14. 2	15. 3
16. 3	17. 3	18. 1	19. 4	20. 1
21. 1	22. 1	23. 1	24. 3	25. 4
26. 3	27. 3	28. 4	29. 4	30. 4
31. 1	32. 3	33. 1	34. 4	35. 4
36. 1	37. 1	38. 2	39. 4	40. 4
41. 3	42. 1	43. 2	44. 3	45. 3

HINTS & SOLUTIONS

1.Sol: Let the point (x_1,y_1) be on the line

i.e., $$3x_1+4y_1=5 \quad (1)$$

Also, which is equilateral from (1, 2) and (3, 4)

i.e., $$(x_1-1)^2+(y_1-2)^2=(x_1-3)^2+(y_1-4)^2$$

simplying, we egt

$$4x_1+4y_1=20 \quad (2)$$

on solving Eqs. (1) and (2), we get

$$x_1=15, y_1=-10$$

2.Sol: Given, $2a+b+3c=0$ (1)

i.e., $$c=\frac{-(2a+b)}{3}$$

put c in $ax+by+c=0$

i.e., $ax+by-\frac{2}{3}a-\frac{b}{3}=0$

i.e., $a\left(x-\frac{2}{3}\right)+b\left(y-\frac{1}{3}\right)=0$

comparing coefficients of line terms

i.e., $x-\frac{2}{3}=0$ and $y-\frac{1}{3}=0$

$\Rightarrow$ $$x=\frac{2}{3}, y=\frac{1}{3}$$

3.Sol: According to question,

$$\{x-(a+b)\}^2+\{y-(b-a)\}^2$$
$$=\{x-(a-b)\}^2+\{y-(a+b)\}^2$$

$$\Rightarrow \quad x^2+(a+b)^2-2x(a+b)+y^2 +(b-a)^2-2y(b-a)$$

$$=x^2+(a-b)^2-2x(a-b)+y^2 +(a+b)^2-2y(a+b)$$

$$\Rightarrow \quad bx-ay=0$$

4.Sol: Given equation of lines are

$$\sqrt{3}x+y=1 \qquad (1)$$

and $x+\sqrt{3}y=1$ (2)

Let m_1 and m_2 be slopes of (1) and (2) then

$$m_1=-\sqrt{3}, m_2=-\frac{1}{\sqrt{3}}$$

Let θ be angle between them

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{-\sqrt{3}+\frac{1}{\sqrt{3}}}{1+\sqrt{3}\cdot\frac{1}{\sqrt{3}}}\right|$$

$$=\left|\frac{-3+1}{2\sqrt{3}}\right|=\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \theta=\frac{\pi}{6}$$

5.Sol: Given equation $x^2-4xy+3y^2=0$

$\therefore$ $m_1+m_2=\frac{4}{3}$ and $m_1m_2=\frac{1}{3}$

On solving these equations, we get

$$m_1=1, m_2=\frac{1}{3}$$

Let the lines parallel to given line are

$y=m_1x+c_1$ and $y=m_2x+c_2$

$\therefore$ $y=\frac{1}{3}x+c_1$ and $y=x+c_2$

Also, these lines passes through the point $(3,-2)$

$$\therefore \quad -2=\frac{1}{3}\times 3+c_1$$

$$\Rightarrow \quad c_1=-3$$

and $\quad -2=1\times 3+c_2$

$$\Rightarrow \quad c_2=-5$$

$\therefore$ Required equation of pair of lines is

$$(3y-x+9)(y-x+5)=0$$

$$\Rightarrow x^2+3y^2-4xy-14x+24y+45=0$$

6.Sol: Equation of line perpendicular to $2x+y+6=0$ and passes through origin is $x-2y=0$

Now, the point of intersection of $2x+y+6=0$ and $x-2y=0$ is $\left(-\frac{12}{5},-\frac{6}{5}\right)$

Similarly, the point of intersection of $x-2y=0$ and $4x+2y-9=0$ is $\left(\frac{9}{5},\frac{9}{10}\right)$

Let the origin divide the line $x-2y=0$ in the ratio $\lambda:1$

That is using section rule, we get

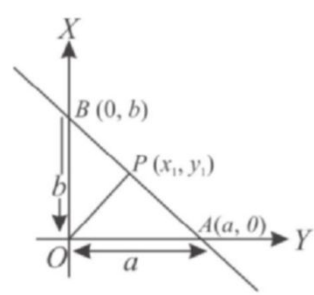
$$\therefore \quad x=\frac{\frac{9}{5}\lambda-\frac{12}{5}}{\lambda+1}=0$$

$$\Rightarrow \quad \frac{9}{5}\lambda=\frac{12}{5}$$

$$\Rightarrow \quad \lambda=\frac{12}{9}=\frac{4}{3}$$

7.Sol: Equation of line is $\frac{x}{a}+\frac{y}{b}=1$ (1)

Let the foot of the perpendicular drawn from the origin to the line be $P(x_1,y_1)$



Any perpendicular line through origin to the given line is drawn, such that product of their slope is -1.

i.e., $$\frac{y_1}{x_1}\times\frac{b}{-a}=-1\Rightarrow by_1=ax_1 \quad (2)$$

Since, P lies on the line AB, so

$$\frac{x_1}{a}+\frac{y_1}{b}=1$$

$$\Rightarrow \quad bx_1+ay_1=ab \quad (3)$$

From (2) and (3), we get

$$x_1=\frac{ab^2}{a^2+b^2} \text{ and } y_1=\frac{a^2b}{a^2+b^2}$$

Now, $$x_1^2+y_1^2=\left(\frac{ab^2}{a^2+b^2}\right)^2+\left(\frac{a^2b}{a^2+b^2}\right)^2$$

$$=\frac{a^2b^4}{(a^2+b^2)^2}+\frac{a^4b^2}{(a^2+b^2)^2}$$

$$=\frac{a^2b^2}{(a^2+b^2)^2}(a^2+b^2)$$

$$=\frac{a^2b^2}{a^2+b^2}=\frac{1}{\frac{1}{a^2}+\frac{1}{b^2}}$$

$$\Rightarrow \quad x_1^2+y_1^2=\frac{1}{\left(\frac{1}{c^2}\right)} \quad \left(\because \frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}\right)$$

Thus, the locus of $P(x_1,y_1)$ is $x^2+y^2=c^2$ which is a circle.

8.Sol: Diagonals of parallelogram bisects each other. That is diagonal BD passes through mid point of AC.

so midpoint of BD is $\left(\frac{5}{2},\frac{7}{2}\right)$

$\therefore$ Equation of diagonal passing through (5, 1) and $\left(\frac{5}{2},\frac{7}{2}\right)$ is

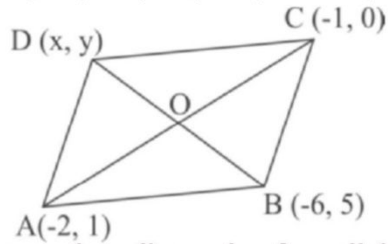
$$y-1=\frac{\frac{7}{2}-1}{\frac{5}{2}-5}(x-5)$$

$$\Rightarrow \quad y-1=-1(x-5)$$

$$\Rightarrow \quad y+x-6=0$$

9.Sol: Since the given points are extremities of a diagonal, that is vertices of a parallelogram.

$A(-2,1), B(-6,5)$. $C(-1,0)$ and $D(x,y)$.



We know that, diagonals of parallelogram bisect each other. We have

$$\left(\frac{-2-1}{2},\frac{1+0}{2}\right)=\left(\frac{x-6}{2},\frac{y+5}{2}\right)$$

$$\Rightarrow \quad \frac{-3}{2}=\frac{x-6}{2} \text{ and } \frac{1}{2}=\frac{y+5}{2}$$

$$\Rightarrow \quad x=-3+6 \text{ and } y=1-5$$

$$\Rightarrow \quad x=3 \text{ and } y=-4$$

Thus, coordinates of fourth vertex of parallelogram are $(3,-4)$.

10.Sol: Given pair of straight lines is rewritten as $x^2-2xy-xy+2y^2=0$

$$(x-2y)(x-y)=0$$

i.e., $$x=2y, x=y \quad (1)$$

Also, $$x+y+1=0 \quad (2)$$

On solving eqs (1) and (2), we get

$$A\left(-\frac{2}{3},-\frac{1}{3}\right), B\left(-\frac{1}{2},-\frac{1}{2}\right), C(0,0)$$

$$\therefore \text{ Area of } \Delta ABC=\frac{1}{2}\begin{vmatrix}-\frac{2}{3} & -\frac{1}{3} & 1\\ -\frac{1}{2} & -\frac{1}{2} & 1\\ 0 & 0 & 1\end{vmatrix}$$

$$=\frac{1}{2}\left[\frac{1}{3}-\frac{1}{6}\right]=\frac{1}{2}\left[\frac{1}{6}\right]=\frac{1}{12}$$

11.Sol: Area of $\Delta=\frac{1}{2}\left\|\begin{matrix}0 & 0 & 1\\ a\cos\theta & b\sin\theta & 1\\ a\cos\theta & -b\sin\theta & 1\end{matrix}\right\|$

$$\Rightarrow \quad \Delta=\frac{1}{2}\left|[1(-ab\sin\theta\cos\theta - ab\sin\theta\cos\theta)\right|$$

$$=\frac{ab\sin 2\theta}{2}$$

Since, maximum value of $\sin 2\theta$ 1, when $\theta=\frac{\pi}{4}$

$$\therefore \quad \Delta_{max}=\frac{ab}{2}$$

12.Sol: Given lines are

$$xy+2x+2y+4=0 \quad (1)$$

and $\quad x+y+2=0 \quad (2)$

rewriting the eq (1), we get

$$(x+2)(y+2)=0$$

i.e., $\quad x=-2$ and $y=-2$

Now the coordinates of vertices are $(-2,0),(0,-2),(-2,-2)$ which is forming a right angled triangle $(-2,-2)$

$\therefore$ circumcenter of the triangle is mid point of the hypotenuse $(-1,-1)$.

13.Sol: SInce, the third vertex (x_1,y_1) lie on the line $x+y=5$.

i.e., $\quad x_1+y_1=5$

$\Rightarrow \quad y_1=5-x_1$

$\therefore$ Coordinate of C is $(x_1,5-x_1)$.

Given, area of $\Delta ABC=4$ units

$$\therefore \quad \frac{1}{2}\begin{vmatrix}2 & -1 & 1\\ 3 & 2 & 1\\ x_1 & 5-x_1 & 1\end{vmatrix}=4$$

using $R_2\to R_2-R_1$ and $R_3\to R_3-R_1$,

$$\begin{vmatrix}2 & -1 & 1\\ 1 & 3 & 0\\ x_1-2 & 6-x_1 & 0\end{vmatrix}=8$$

$\Rightarrow \quad 6-x_1-3(x_1-2)=\pm 8$

$\Rightarrow \quad 6-x_1-3x_1+6=\pm 8$

$\Rightarrow \quad 12-8=4x_1$ or $4x_1=20$

$\Rightarrow \quad x_1=1$ or $x_1=5$

$\therefore \quad y_1=5-1=4$ or $y_1=0$

$\therefore \quad C(x_1,y_1)=C(1,4)$ or $C(5,0)$

14.Sol: $2x^2-5xy+2y^2+3x-3y+1=0$

$\Rightarrow \quad (x-2y+1)(2x-y+1)=0$

$\therefore$ Two equations are $x-2y+1=0$ and $2x-y+1=0$.

Length of perpendicular from $(-1,2)$ are

$$p_1=\left|\frac{-1-4+1}{\sqrt{1+4}}\right|=\frac{4}{\sqrt{5}} \text{ and}$$

$$p_2=\left|\frac{-2-2+1}{\sqrt{5}}\right|=\frac{3}{\sqrt{5}}$$

$\therefore$ Product $=p_1\cdot p_2=\frac{12}{5}$

15.Sol: Given pair of lines be

$$kx^2+5xy+y^2=0 \quad (1)$$

On comparing eq (1) with $ax^2+2hxy+by^2=0$, we get

$a=k, b=1$ and $2h=5$

Let m_1 and m_2 be two slopes of pair of lines.

i.e., $\quad m_1+m_2=\frac{-2h}{b}=-5$

and $\quad m_1m_2 = \frac{a}{b} = k$

Now, $\quad (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2$

$\Rightarrow \quad (1)^2 = (-5)^2 - 4k$

[given, $m_1 - m_2 = 1$ or $m_2 - m_1 = 1$]

$\Rightarrow \quad 1 = 25 - 4k$

$\Rightarrow \quad 4k = 24 \Rightarrow k = 6$

16.Sol: The second degree equation will represent a pair of perpendicular straight lines, if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \text{ and } a + b = 0$$

Given pair of line is

$12x^2 + 7xy - py^2 - 18x + qy + 6 = 0$

$$\therefore \quad \begin{vmatrix} 12 & 7/2 & -9 \\ 7/2 & -p & q/2 \\ -9 & q/2 & 6 \end{vmatrix} = 0$$

and $\quad 12 - p = 0 \Rightarrow p = 12$

$$\therefore \quad \begin{vmatrix} 12 & 7/2 & -9 \\ 7/2 & -p & q/2 \\ -9 & q/2 & 6 \end{vmatrix} = 0$$

$$\Rightarrow \quad 12\left(-72 - \frac{q^2}{4}\right) - \frac{7}{2}\left(21 + \frac{9q}{2}\right)$$

$$-9\left(\frac{7q}{4} - 108\right) = 0$$

$$\Rightarrow \quad -864 - 3q^2 - \frac{147}{2} - \frac{63q}{4}$$

$$-\frac{63q}{4} + 972 = 0 \Rightarrow q = 1$$

17.Sol: Comparing the given equation with standrad equation, we get $a = 12$ and $b = a$, for perpendicular lines

coefficient of x^2 + coefficients of $y^2 = 0$

$\therefore \quad 12 + a = 0$

$\Rightarrow \quad a = -12$

18.Sol: Here, $a = 2, b = 3, h = \frac{5}{2}$

$$\because \quad \tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$\therefore \quad \tan\theta = \frac{2\sqrt{\left(\frac{5}{2}\right)^2 - 6}}{2 + 3}$$

$$\Rightarrow \quad m = \frac{1}{5}$$

19.Sol: The given equation is

$$4x^2 + hxy + y^2 = 0 \qquad (1)$$

This equation represent coincident lines, if

$\because \quad h^2 = ab$

$$\left(\frac{h}{2}\right)^2 = 4(1)$$

$h^2 = 16$

$h = 4$

20.Sol: Given equations of lines are

$3x - 4y + 7 = 0$

and $\quad -12x - 5y + 2 = 0$

we have $a_1a_2 + b_1b_2 = 3 \times (-12) + (-4)(-5)$

$= -36 + 20 = -16$

$\Rightarrow \quad a_1a_2 + b_1b_2 \le 0$

$\therefore$ Obtuse angle bisector is

$$\frac{3x - 4y + 7}{\sqrt{3^2 + (-4)^2}} = -\frac{-12x - 5y + 2}{\sqrt{(-12)^2 + (-5)^2}}$$

$\Rightarrow \quad 13(3x - 4y + 7) = -5(-12x - 5y + 2)$

$\Rightarrow \quad 21x + 77y - 101 = 0$

21.Sol: We have, $ax^2 + 2hxy + by^2 = 0$

Let slope of one line is m

$\therefore$ Slope of another line is $2m$.

we know that, $m_1 + m_2 = -\frac{2h}{b}$

and $\quad m_1 m_2 = \frac{a}{b}$

$\therefore \quad m + 2m = -\frac{2h}{b}$

$\Rightarrow \quad 3m = \frac{-2h}{b}$

and $\quad m(2m) = \frac{a}{b}$

$\Rightarrow \quad 2m^2 = \frac{a}{b}$

On eliminating m, we get

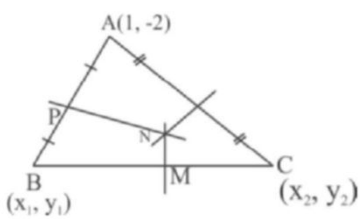
$$2\left(\frac{-2h}{3b}\right)^2 = \frac{a}{b}$$

$\Rightarrow \quad 8h^2 = 9ab$

22.Sol: Let $B(x_1, y_1)$ and $C(x_2, y_2)$ be two vertices and $P\left(\frac{x_1+1}{2}, \frac{y_1-2}{2}\right)$ lies on perpendicular bisector $x - y + 5 = 0$

$\therefore \quad \frac{x_1+1}{2} - \frac{y_1-2}{2} = -5$

$\Rightarrow \quad x_1 - y_1 = -13 \quad (1)$



Also, PN is perpendicular to AB

$\therefore \quad \frac{y_1+2}{x_1-1} \times 1 = -1$

$\Rightarrow \quad y_1 + 2 = -x_1 + 1$

$\Rightarrow \quad x_1 + y_1 = -1 \quad (2)$

On solving eqs (1) and (2), we get

$x_1 = -7, y_1 = 6$

$\therefore$ The coordinates of B are $(-7, 6)$. Similarly, the coordinates of C are $\left(\frac{11}{5}, \frac{2}{5}\right)$.

Hence, the equation of BC is

$$y - 6 = \frac{\frac{2}{5} - 6}{\frac{11}{5} + 7}(x + 7)$$

$\Rightarrow \quad y - 6 = \frac{-14}{23}(x + 7)$

$\Rightarrow \quad 14x + 23y - 40 = 0$

23.Sol: Given equation is

$x^2 - xy - 6y^2 - 7x + 31y - 18 = 0$

Here, $a = 1, b = -6, h = \frac{-1}{2}$

$$\therefore \quad \theta = \tan^{-1}\left|\frac{2\sqrt{\left(-\frac{1}{2}\right)^2 - 1\times(-6)}}{1 + (-6)}\right|$$

$$= \tan^{-1}\left|\frac{2\sqrt{\frac{1}{4} + 6}}{-5}\right|$$

$$= \tan^{-1}|-1|$$

$$= \frac{\pi}{4}$$

24.Sol: Given points will be collinear, if

$$\begin{vmatrix} a & b & 1 \\ b & a & 1 \\ a^2 & -b^2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & b & 1 \\ b-a & a-b & 0 \\ a^2-a & -b^2-b & 0 \end{vmatrix} = 0$$

[applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$]

$$\Rightarrow (a-b)\begin{vmatrix} a & b & 1 \\ -1 & 1 & 0 \\ a^2-a & -b^2-b & 0 \end{vmatrix} = 0$$

$$\Rightarrow (a-b)(b^2+b-a^2+a) = 0$$

$$\Rightarrow (a-b)\{(a+b)-(a^2-b^2)\} = 0$$

$$\Rightarrow (a-b)(a+b)(1-a+b) = 0$$

$$\Rightarrow a = b \text{ or } a+b=0 \text{ or } a = 1+b$$

25.Sol: Given equation is

$2x^2 + 3xy - 5y^2 = 0$

On comparing with

$ax^2 + 2hxy + by^2 = 0,$ we get

$$a = 2, h = \frac{3}{2}, b = -5$$

$$\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$$

$$= \frac{2\sqrt{\left(\frac{3}{2}\right)^2+10}}{2-5}$$

$$= \frac{2\sqrt{\frac{9}{4}+10}}{-3}$$

$$= \frac{\sqrt{49}}{-3}$$

$$\Rightarrow \theta = \tan^{-1}\left(-\frac{7}{3}\right)$$

26.Sol: Given equation of straight line is

$x^2 - 5xy + 4y^2 + 3x - 4 = 0$

$$\therefore \tan\theta = \left|\frac{2\sqrt{\left(-\frac{5}{2}\right)^2 - 4}}{5}\right|$$

$$= \left|\frac{2\sqrt{\frac{25}{4}-4}}{5}\right| = \frac{2}{5}\times\sqrt{\frac{9}{4}} = \frac{2}{5}\times\frac{3}{2} = \frac{3}{5}$$

$$\Rightarrow \tan^2\theta = \frac{9}{25}$$

27.Sol: From the given equation,

$$3(my+2) + 2y = 10$$

$$\Rightarrow y(3m+2) = 4$$

$$\Rightarrow y = \frac{4}{3m+2}$$

since, $y \in z \Rightarrow 3m+2$

$\Rightarrow \pm1, \pm2, \pm4$

$$\Rightarrow m = -1, -\frac{1}{3}, -\frac{4}{3}, 0, \frac{2}{3}, -2$$

28.Sol: Area of triangle having vertices

$(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is given by

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\therefore \text{Required area} = \frac{1}{2}\begin{vmatrix} -2 & 2 & 1 \\ 1 & 5 & 1 \\ 6 & -1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[-2(5+1) - 2(1-6) + 1(-1-30)\right]$$

$$= \frac{1}{2}\left[-12+10-31\right]$$

$$=\frac{-33}{2}$$

$\therefore$ Area $=\frac{33}{2}$ sq units

29.Sol: Equation of bisectors of the angle between

$$x^2-2nxy-y^2=0$$

$$\Rightarrow \quad x^2+\frac{2}{n}xy-y^2=0$$

This equation is identical to

$$x^2-2mxy-y^2=0$$

$$\therefore \quad \frac{2/n}{-2m}=1$$

$$\Rightarrow \quad \frac{1}{n}=-m$$

$$\Rightarrow \quad nm+1=0$$

30.Sol: Let $A(1,0),B(0,1)$ and $C(x,8)$

Since, A,B and C are collinear, then slope of $AB=$ Slope of BC

$$\Rightarrow \quad \frac{1-0}{0-1}=\frac{8-1}{x-0}$$

$$\Rightarrow \quad -1=\frac{7}{x}$$

$$\Rightarrow \quad x=-7$$

31.Sol: Let the equation of the line be

$$\frac{x}{a}+\frac{y}{b}=1 \qquad (1)$$

which passes through (4, 3).

$$\therefore \quad \frac{4}{a}+\frac{3}{b}=1 \qquad (2)$$

Its is given that $a+b=-1$ (3)

On solving eqs. (1) and (2), we get

$a=-2,b=1$ or $a=2,b=-3$

On substituting the values of a and b in eq. (1), we get

$$\frac{x}{-2}+\frac{y}{1}=1 \text{ and } \frac{x}{2}+\frac{y}{-3}=1$$

32.Sol: Here, $x_1=4,y_1=-2,x_2=8,y_2=6$ and $m:n=7:5$

$$\therefore \quad x=\frac{mx_2+nx_1}{m+n}=\frac{7\times8+5\times4}{12}$$

$$=\frac{56+20}{12}=\frac{76}{12}=\frac{19}{3}$$

and $$y=\frac{my_2+ny_1}{m+n}$$

$$=\frac{7\times6+5\times(-2)}{7+5}$$

$$=\frac{42-10}{12}=\frac{32}{12}=\frac{8}{3}$$

$$\therefore \quad (x,y)=\left(\frac{19}{3},\frac{8}{3}\right)$$

33.Sol: The equation of a line through $P(1,1)$ and parallel to $x+y=1$, is

$$\frac{x-1}{\cos\frac{3\pi}{4}}=\frac{y-1}{\sin\frac{3\pi}{4}}$$

Let $PM=r$

Then, the coordinates of M given by

$$\frac{x-1}{\cos\frac{3\pi}{4}}=\frac{y-1}{\sin\frac{3\pi}{4}}=r \text{ is } \left(1-\frac{r}{\sqrt{2}},1+\frac{r}{\sqrt{2}}\right)$$

$\because$ M lies on $2x-3y-4=0$

$$\therefore \quad 2\left(1-\frac{r}{\sqrt{2}}\right)-3\left(1+\frac{r}{\sqrt{2}}\right)-4=0$$

$$\Rightarrow \quad 2-\frac{2r}{\sqrt{2}}-3-\frac{3r}{\sqrt{2}}-4=0$$

$$\Rightarrow \quad -5-\frac{5r}{\sqrt{2}}=0$$

$$\Rightarrow \quad r=-\sqrt{2}$$

Hence, $PM=\sqrt{2}$

34.Sol: Area of triangle $=\frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$

$$=\frac{1}{2}\begin{vmatrix} a & a+b+c & 1 \\ b & a+b+c & 1 \\ c & a+b+c & 1 \end{vmatrix}$$

[Applying $c_2 \to c_2 + c_1$]

$$=\frac{a+b+c}{2}\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix}=0$$

35.Sol: Given, equation of line is $4x - ay = 8$ (1)

We can write eq. (1) in intercept form as,

$$\frac{x}{2}+\frac{y}{-\left(\frac{8}{a}\right)}=1$$

Hence, x-intercept $= 2$

and y-intercept $= -\frac{8}{a}$

According to question,

$$-\frac{8}{a}=3\times 2 \Rightarrow -8=6a$$

$$\Rightarrow \qquad a=\frac{-8}{6}=-\frac{4}{3}$$

36.Sol: As the slope of incident ray is $-\frac{1}{\sqrt{3}}$ so the slope of reflected ray has to be $\frac{1}{\sqrt{3}}$.

The point of incidence is $(\sqrt{3},0)$. Hence the equation of reflected ray is $y=\frac{1}{\sqrt{3}}(x-\sqrt{3})$.

$$\therefore \qquad \sqrt{3}y - x = -\sqrt{3}.$$

$$\therefore \qquad x-\sqrt{3}y-\sqrt{3}=0$$

37.Sol: The equation of any line through the point of intersection of the lines $x-y-1=0$ and $2x-3y+1=0$ is

$$(x-y-1)+\lambda(2x-3y+1)=0$$

$$\Rightarrow (2\lambda+1)x-y(3\lambda+1)+(\lambda-1)=0 \qquad (1)$$

The line in eq. (1) will be parallel to x-axis, if it is of the form $y=$ constant, therefore coefficient of x in eq. (1) $=0$

i.e., $\qquad 2\lambda+1=0 \Rightarrow \lambda=-\frac{1}{2}$

On putting $\lambda=-\frac{1}{2}$ in eq. (1), we get $y=3$

This is the equation of the required line.

38.Sol: Given equations of line is $\frac{x}{a}+\frac{y}{b}=1$ (1)

$$\Rightarrow \qquad bx+ay=ab$$

$$\Rightarrow \qquad bx+ay-ab=0$$

$$\therefore \qquad m=-\frac{b}{a}$$

So, equation of line passing through (a,b) and parallel to eq. (1) is

$$y-b=-\frac{b}{a}(x-a)$$

$$ay-ab=-bx+ab$$

$$ay+bx=2ab$$

$$\frac{y}{b}+\frac{x}{a}=2$$

$$\Rightarrow \qquad \frac{x}{a}+\frac{y}{b}=2$$

39.Sol: $-\frac{4p}{8}=-\frac{3}{8}\Rightarrow p=\frac{3}{4}$

40.Sol: We have, $y=2x+1, y=3x+1, x=4$

Intersecting points of above lines are

$\therefore$ Area of triangle

$$=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_1 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$=\frac{1}{2}\begin{vmatrix} 0 & 1 & 1 \\ 4 & 9 & 1 \\ 4 & 13 & 1 \end{vmatrix}$$

$$=\frac{1}{2}\left[0(9-13)-1(4-4)+1(52-36)\right]$$

$$=\frac{1}{2}\times 16=8$$

41.Sol: The given equation of pair of straight lines can be rewritten as $\left(\sqrt{3}x-y\right)\left(x-\sqrt{3}y\right)=0.$

Their seperate equations are.

$y=\sqrt{3}x$ and $y=\frac{1}{\sqrt{3}}x$

$\Rightarrow \quad y=\tan 60°x$ and $y=\tan 30°x$

After rotation, the separate equation are

$y=\tan 60°x$

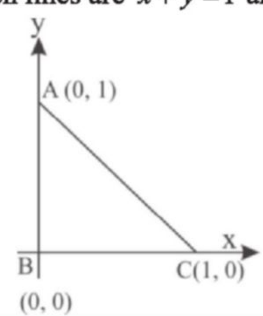
and $\quad y=\tan 30°x$

$\Rightarrow \quad x=0$ and $y=\sqrt{3}\cdot x$

$\therefore$ combined equation in the new position is

$x\left(\sqrt{3}x-y\right)=0$ or $\sqrt{3}x^2-xy=0$

42.Sol: Given lines are $x+y=1$ and $xy=0$



$xy=0$ represents line $x=0$ and $y=0$

Triangle formed by lines $x+y=1, x=0$ and $y=0$ is ΔABC.

$\because$ ΔABC is right angled triangle at $\angle B$.

Orthocentre of ΔABC is at $B(0,0)$

43.Sol: Given equation of pair of straight lines is

$x^2+y^2+2gx+2fy+1=0$

Since, the necessary and sufficient condition for pair of straight lines is

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} 1 & 0 & g \\ 0 & 1 & f \\ g & f & 1 \end{vmatrix}=0$$

$$\Rightarrow \quad 1\left(1-f^2\right)+g\left(0-g\right)=0$$

$$\Rightarrow \quad 1-f^2-g^2=0$$

$$\Rightarrow \quad f^2+g^2=1$$

44.Sol: Conceptual

45.Sol: We have, $\frac{x}{10}-\frac{y}{4}=3$

$$\Rightarrow \quad 2x-5y=60$$

$$\therefore \text{ Slope}=\frac{-a}{b}$$

$$=\frac{2}{5}$$

NTSE

FUNDAMENTAL THEOREM

[2018-2019]

1. If $x^m.y^n=7889$, where and are prime numbers then value of $x+y$ is $\cdots$ **[Chandigarh]**
(a) 30 (b) 60 (c) 100 (d) 300

2. $G.C.D$ of 4 and 19 is$\cdots$ **[Gujarat]**
(a)1 (b) 4 (c) 19 (d) 76

3. The $H.C.F$ and $L.C.M$ of two numbers are 12 and 240 respectively. If one of these number is 48 what the others numbers will be ? **[Uttarakhand]**
(a) 58 (b) 60 (c) 70 (d) 80

4. If the $L.C.M$ of 12 and 42 is then the value of *'m'* is **[Tamil Nadu]**
(a) 50 (b) 8 (c) $\frac{1}{5}$ (d) 1

5. A positive integer n has 60 divisors and $7n$ has 80 divisors. What is the greatest integer k such that divides n ? **[Telangana]**
(a) 0 (b) 1 (c) 2 (d) 3

6. What is the largest integer that is a divisor of
$(n+1)(n+3)(n+5)(n+7)(n+9)$
for all positive even integer n ? **[Telangana]**
(a) 3 (b) 5 (c) 11 (d) 15

7. The sum of 18 consecutive positive integers is a perfect square. The small possible value of this sum is **[Telangana]**
(a) 169 (b) 225 (c)289 (d) 361

[2015-2016]

8. For how many values of n (Where n is an integer), the expression $\frac{8(n^3-3n^2+5)}{2n-1}$ is an integer ? **[Punjab]**
(a)8 (b)4 (c) 11 (d) 28

9. $N=69^5+5\cdot 69^4+10\cdot 69^3+5\cdot 69+1.$ How many positive intergers are of N ? **[Telangana]**
(a) 5 (b) 216 (c) 69 (d) 125

10. For how many integer n, is $\frac{n}{20-n}$ the square of an integer ? **[Telangana]**
(a) 1 (b)12 (c) 3 (d)4

11. If a number is divided by 6, the remainder is 3 then what will be the remaimder when the square of the same number is divided by 6 again **[Uttar pradesh]**
(a) 0 (b) 1 (c) 12 (d) 3

12. The HCF of two expressions p and q is 1. Their LCM is : **[Chandigarh]**
(a) $p+q$ (b) $p-q$ (c) pq (d) $\frac{1}{pq}$

13. If the $L.C.M$ of two prime numbers is 2520 and $H.C.F$ is 12. Its one number is 504, then the other number will be **[Madhya Pradesh]**
(a) 50 (b) 65 (c) 40 (d) 60

14. The HCF of any two prime numbers a and b, is **[Rajasthan]**
(a) a (b) ab (c) b (d)1

15. The unit digit in the decimal expansion of is : **[Tamil Nadu]**
(a) 1 (b) 3 (c) 5 (d) 7

16. If $G.C.D$ of two numbers is 8 and their product is 384, then their $L.C.M$ is **[Gujarat]**
(a) 24 (b) 16 (c) 32 (d) 48

17. The G.C.D of the numbers $2^{100}-1$ and $2^{120}-1$ is **[Andhra pradesh]**
(a) $2^{60}-1$ (b) $2^{20}-1$ (c) 2^{20} (d) 2^{10}

ANSWER KEY

1. c	2. a	3. b	4. b	5. c
6. d	7. b	8. a	9. b	10. d
11. d	12. c	13. d	14. d	15. d
16. d	17. b			

HINTS & SOLUTIONS

1.Sol: Given that $x^m y^n = 7889$. Since x and y are prime numbers. Then factorising R.H.S , we get

$$x^m y^n = 7^3\ 23^1$$

$\therefore \quad x = 7$ and $y = 23$

Hence $\quad x+y = 7+23=30$

2.Sol: Since 4 is even and 19 is odd Therefore only one common factor exists. That would be the $G.C.D$

3.Sol: As we know, the product of any two integers is equal to the product of their LCM and GCD.

Let the missing number be k. solving the following equation

$$240\times 12 = 48\times k$$

i e $\quad k=\dfrac{240\times 12}{48} = 60$

4.Sol: L.C.M of 12 and 42 is 84. That is

$$10m+4=84$$

i.e., $\quad 10m = 80$

$$m = 8$$

5.Sol: Let the prime factors of n be

$p_1, p_2, p_3, \text{---}\ p_r$ and 7, where $p_i \neq 7$

i e $n = p_1^{a_1}\times p_2^{a_1}\times p_3^{a_3}\times \text{---}\ p_r^{ar}\times 7^k.$

Also given that number of divisors of n is 60.

That is $(a_1+1)(a_2+1)(a_3+1)$

$$\ldots(a_r+1)(k+1) = 60 \qquad (1)$$

also $\quad 7n = P_a^{a_1} * P_2^{a_2} * P_3^{a_3} * \ldots * P_r^{ar} * 7^{k+1}$

Number of divisors of $7n$ is 80.

That is $(a_1+1)(a_2+1)(a_3+1)\ldots$

$$(a_r+1)(k+2) = 80 \qquad (2)$$

Form (1) and (2), we get

$$\frac{(a_1+1)(a_2+1)(a_3+1)\ldots(a_r+1)(k+2)}{(a_1+1)(a_2+1)(a_3+1)\ldots(a_r+1)(k+1)} = \frac{80}{60}$$

i.e., $\quad \dfrac{k+2}{k+2} = \dfrac{4}{3}$

$\therefore \quad k = 2$

6.Sol: Let $p(n) = (n+1)(n+3)(n+5)$

$$(n+7)(n+9)$$

since n is even, so $(n+1),(n+3),(n+5)$ $(n+7)$ and $(n+9)$ are five consecuitive odd numbers. That is $p(n)$ is a multiple of 3 and 5, so $p(n)$ is divisible by 15, which is the largest integer divisor.

7.Sol: Denoting the first term of the sequence by a then it is arithmetic sequence with common difference $d=1$, has the sum,

$$a+(a+1)+\ldots+(a+17) = 18(a)$$

$$+(1+2+\ldots+17)$$

$$= 18\,a + \frac{1}{2} + (17\times 18)$$

$$= 9(2a+17)$$

For the sum to be a perfect square, the term $2a+17$ must be perfect square, by inspection, we can see that this first occurs when $a = 4$, This gives the minimal sum, which is $9\times 25 = 225$

8.Sol: Given that $\dfrac{8(n^3-3n^2+5)}{2n-1}$

Rewritting above expressions as

$$4n^2 - 10n - 5 + \frac{35}{2n-1}$$

clearly $4n^2-10n-5$ is an integer for all $n \in z$

so that order to get an integer $\frac{35}{2n-1}$

must be an integer. It is easy to verify that

$\frac{35}{2n-1}$ is an integer when $n \in \{-17,-3,-2,0,$

$n \in 1,3,4,18\}$ For $n \geq 9$ and $n \leq -17,(2n-1)$

Can't divide 35.

9.Sol: This indeed the binomial expansion of

$(69+1)^5$ so

$70^5=(2\times5\times7)^5$

$=2^5\times5^5\times7^5$

so the total number of factors is

$(5+1)\times(5+1)(5+1)=216.$

10.Sol: **Method 1.** Let $x^2=\frac{n}{20-n},$ with $x \geq 0$

(note that the solutions $x<0$ do not give any additional soultions for n). Then rewriting,

$n=\frac{20x^2}{x^2+1}$. Since $gcd(x^2,x^2+1)=1$, it follows

that x^2+1 divided 20. Listing the factors of 20, we find that $x=0, 1, 2, 3$ are the only (D) solutions (respectively yielding $n=0,10,16,18$).

Method 2. If $\frac{n}{20-n}=k^2 \geq 0,$ then $n \geq 0$ and

$20-n>0,$ otherwise $\frac{n}{20-n}$ will be

negative. Thus $0 \leq n \leq 19$ and

$$0=\frac{0}{20-(0)} \leq \frac{n}{20-n} \leq \frac{19}{20-(19)}=19$$

Checking all k for which $0 \leq k^2 \leq 19,$ we have 0, 1, 2, 3 as the possibilities. (D)

11.Sol: Let the number be x.

$$x=6q+3$$

Squaring both sides, we get

$$x^2=36q^2+9+36q=36(q^2+q)+6+3$$

Thus, when square of the given number is divided by 6 then remainder is 3.

12.Sol: The product of two polynomials is equal to the product of their *HCF* and *LCM*

we have *HCF* $\times$ *LCM* = pq

Given *HCF* = 1,So *LCM=pq*

13.Sol: Given that LCM of two numbers is 2520 and HCF is 12. Let the other number is *x*.

we have $2520\times12=504\times x$

$$\Rightarrow x=\frac{30240}{504}$$

$$=60$$

14.Sol: Given two prime numbers *a* and *b* then HCF of *a* and *b* is 1

15.Sol: We know that

$7^1=7, 7^2=49, 7^3=343, 7^4=2401$

$7^5=16807, 7^6=117649$

$\therefore$ Unit digit in the expansions of 7^{25} ie.,$(7^5)^5$

Will be 7.

16.Sol: We have product of GCD and LCM is equal to product of two numbers

i.e., 8 $\times$ LCM =384

$$\therefore \text{LCM}=\frac{384}{8}=48$$

17.Sol: We know a^n-b^n is always divisible by *a -b* whether n is even or odd.

2^{120}-1=$(2^{20})^6$-1^6 is divisible by $2^{20}-1$

2^{100-1}=$(2^{20})^5$-1^5 is divisible by $2^{20}-1$

$\therefore$ GCD is $2^{20}-1$

SOLVED PAPER

KVPY 2019

★ Mathematics ★

KVPY - SX

1. The number of four-letter words that can be formed with letters a, b, c such that all three letters occur is
(a) 30 (b) 36 (c) 81 (d) 256

2. Let $A=\left\{\theta \in R:\left(\frac{1}{3}\sin(\theta)+\frac{2}{3}\cos(\theta)\right)^2 =\frac{1}{3}\sin^2(\theta)+\frac{2}{3}\cos^2(\theta)\right\}$. Then
(a) $A\cap[0,\pi]$ is an empty set
(b) $A\cap[0,\pi]$ has exactly one point
(c) $A\cap[0,\pi]$ has exactly two points
(d) $A\cap[0,\pi]$ has more than two points

3. The area of the region bounded by the lines $x = 1$, $x = 2$, and the curves $x\left(y-e^x\right)=\sin x$ and $2xy = 2\sin x + x^3$ is
(a) $e^2-e-\frac{1}{6}$ (b) $e^2-e-\frac{7}{6}$
(c) $e^2-e+\frac{1}{6}$ (d) $e^2-e+\frac{7}{6}$

4. Let AB be a line segment with midpoint C, and D be the midpoint of AC. Let C_1 be the circle with diameter AB, and C_2 be the circle with diameter AC. Let E be a point of C_1 such that EC is perpendicular to AB. Let F be a point on C_2 such that DF is perpendicular to AB, and E and F lie on opposite sides of AB. Then the value of $\sin\angle FEC$ is
(a) $\frac{1}{\sqrt{10}}$ (b) $\frac{2}{\sqrt{10}}$ (c) $\frac{1}{\sqrt{13}}$ (d) $\frac{2}{\sqrt{13}}$

5. The number of integers x satisfying
$$-3x^4+\det\begin{bmatrix}1 & x & x^2\\ 1 & x^2 & x^4\\ 1 & x^3 & x^6\end{bmatrix}=0$$
(a) 1 (b) 2 (c) 5 (d) 8

6. Let P be anon-zero polynomial such that $P(1 + x) = P(1 - x)$ for all real x, and $P(1) = 0$. Let m be the largest integer such that $(x-1)^m$ divides $P(x)$ for all such $P(x)$. Then m equals
(a) 1 (b) 2 (c) 3 (d) 4

7. Let $f(x) = f(x)=\begin{cases} x\sin\left(\frac{1}{x}\right) & when\ x\neq 0\\ 1 & when\ x=0\end{cases}$

and $A=\{x\in R: f(x)=1\}$. Then A has
(a) Exactly one element
(b) Exactly two elements
(c) Exactly three elements
(d) Infinitely many elements

8. Let S be subset of the plane defined by $S=\{(x,y):|x|+2|y|\}=1$. Then the radius of the smallest circle with centre at the origin and having non-empty intersection with S is

(a) $\frac{1}{5}$ (b) $\frac{1}{\sqrt{5}}$ (c) $\frac{1}{2}$ (d) $\frac{2}{\sqrt{5}}$

9. The number of solutions of the equation $\sin(9x)+\sin(3x)=0$ in the closed interval $[0, 2\pi]$ is

(a) 7 (b) 13 (c) 19 (d) 25

10. Among all the parallelograms whose diagonals are 10 and 4, the one having maximum area has its perimeter lying in the interval

(a) (19, 20] (b) (20, 21]
(c) (21, 22] (d) (22, 23]

11. The number of ordered pairs (a, b) of positive integers such that $\frac{2a-1}{b}$ and $\frac{2b-1}{a}$ are both integers is

(a) 1 (b) 2
(c) 3 (d) more than 3

12. Let $z = x + iy$ and $w = u + iv$ be complex numbers on the unit circle such that $z^2+w^2=1$. Then the number of ordered pairs (z, w) is

(a) 0 (b) 4 (c) 8 (d) infinite

13. Let E denote the set of letters of the English alphabet, $V = \{a, e, i, o, u\}$, and C be the complement of V in E. then the number of four-letter words (where repetitions of letters are allowed) having at least one letter from V and at least one letter from C is

(a) 261870 (b) 314160
(c) 425880 (d) 851760

14. Let $\sigma_1, \sigma_2, \sigma_3$ be planes passing through the origin. Assume that σ_1 is perpendicular to the vector (1, 1, 1), σ_2 is perpendicular to a vector (a, b, c), and σ_3 is perpendicular to thevector (a^2, b^2, c^2). What are all the positive values of a, b, and c so that $\sigma_1 \cap \sigma_2 \cap \sigma_3$ is a single point ?

(a) Any positive value of a, b, and c other than 1
(b) Any positive values of a, b and c where either $a \neq b, b \neq c, a \neq c$
(c) Any three distinct positive values of a, b, and c
(d) There exist no such positive real numbers a, b, and c

15. Ravi and Rashmi are each holding 2 red cards and 2 black cards (all four red and all four black cards are identical). Ravi picks a card at random from Rashmi, and then Rashmi picks a card at random from Ravi. This process is repeated a second time. Let p be the probability that both have all 4 cards of the same colour. Then p satisfies

(a) $p \leq 5\%$ (b) $5\% < p \leq 10\%$
(c) $10\% < p \leq 15\%$ (d) $15\% < p$

16. Let A_1, A_2 and A_3 be the regions on R^2 defined by

$$A_1=\{(x,y): x\geq 0, y\geq 0, 2x+2y-x^2-y^2>1>x+y\},$$

$$A_2=\{(x,y): x\geq 0, y\geq 0, x+y>1>-x^2+y^2\},$$

$$A_3=\{(x,y): x\geq 0, y\geq 0, x+y>1>x^3+y^3\}.$$

Denote by $|A_1|$, $|A_2|$, and $|A_3|$ the areas of the regions A_1, A_2, and A_3 respectively. Then

(a) $|A_1|>|A_2|>|A_3|$ (b) $|A_1|>|A_3|>|A_2|$
(c) $|A_1|=|A_2|<|A_3|$ (d) $|A_1|=|A_3|>|A_2|$

17. Let $f: R \to R$ be a continuous function such that $f(x^2)=f(x^3)$ for all $x \in R$. Consider the following statements

I. f is an odd function
II. f is an even function
III. f is differentiable everywhere

Then

(a) I is true and III is false
(b) II is true and III is false
(c) both I and III are true
(d) both II and III are true

18. Suppose a continuous function $f:[0,\infty)\to R$ satisfies

$$f(x) = 2\int_0^x tf(t)dt \Big| +1 \text{ for all } x \geq 0.$$

Then $f(1)$ equals

(a) e (b) e^2 (c) e^4 (d) e^6

19. Let $a > 0$, $a \neq 1$. Then the set of all positive real numbers b satisfying $(1+a^2)\ (1+b^2) =$ 4ab is

(a) an empty set
(b) a singleton set
(c) a finite set containing more than one element
(d) $(0,\infty)$

20. Let f: $R \to R$ be a function defined by $f(x)$

$$= \begin{cases} \dfrac{\sin(x^2)}{x} & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}.$$

Then, at $x = 0$, f is

(a) Not continuous
(b) Continuous but not differentiable
(c) Differentiable and the derivative is not continuous
(d) Differentiable and the derivative is continuous

21. The points C and D on a semicircle with AB as diameter are such that $AC = 1$, $CD = 2$, and $DB = 3$. Then the length of AB lies in the interval

(a) [4,4.1) (b) [4.1,4.2)
(c) [4.2,4.3) (d) $[4.3,\infty)$

22. Let ABC be a triangle and let D be the midpoint of BC. Suppose $\cot(\angle CAD):\cot(\angle BAD)$ $= 2:1$. If G is the centroid of triangle ABC, then the measure of $\angle BGA$ is

(a) 90° (b) 105° (c) 120° (d) 135°

23. Let $f(x) = x^6 - 2x^5 + x^3 + x^2 - x - 1$ and $g(x) = x^4 - x^3 - x^2 - 1$ be two polynomials. Let a, b, c and d be the roots of $g(x) = 0$. Then the value of $f(a) + f(b) + f(c) + f(d)$ is

(a) –5 (b) 0 (c) 4 (d) 5

24. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = 2\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 5\hat{i} + \hat{j} - \hat{k}$ be the three vectors. The area of the region formed by the set of points whose position vectors $\vec{r}$ satisfy the equations $\vec{r}.\vec{a} = 5$ and $\left|\vec{r} - \vec{b}\right| + \left|\vec{r} - \vec{c}\right| = 4$ is closed to the integer

(a) 4 (b) 9 (c) 14 (d) 19

25. The number of solutions to $\sin\left(\pi\sin^2(\theta)\right)$ $+\sin\left(\pi\cos^2(\theta)\right) = 2\cos\left(\dfrac{\pi}{2}\cos(\theta)\right)$ satisfying $0 \leq \theta \leq 2\pi$ is

(a) 1 (b) 2 (c) 4 (d) 7

26. Let $J = \displaystyle\int_0^1 \frac{x}{1+x^8}dx.$ Conider the following assertions:

I. $J > \dfrac{1}{4}$

II. $J < \dfrac{\pi}{8}$

Then

(a) Only I is true
(b) Only II is true
(c) Both I and II are true
(d) Neither I nor II is true

27. Let $f:(-1,1)\to R$ be a differentiable function satisfying $\left(f'(x)\right)^4 = 16\left(f(x)\right)^2$

for all $x \in (-1,1), f(0)=0$. The number of such funcitons is

(a) 2 (b) 3
(c) 4 (d) More than 4

28. For $x \in R$, let $f(x)=|\sin x|$ and $g(x)=\int_0^x f(t)\,dt$. Let $p(x)=g(x)-\frac{2}{\pi}x$. Then

(a) $p(x+\pi)=p(x)$ for all x

(b) $p(x+\pi)\neq p(x)$ for at least one but finitely many x

(c) $p(x+\pi)\neq p(x)$ for inifinitely many x

(d) p is one-one function

29. Let A be the set of vectors $\vec{a}=(a_1,a_2,a_3)$ satisfying $\left(\sum_{i=1}^{3}\frac{a_i}{2^i}\right)^2=\sum_{i=1}^{3}\frac{a_i^2}{2^i}$. Then

(a) A is empty
(b) A contains exactly one element
(c) A has 6 elements
(d) A has infinitely many elements

30. Let $f:[0,1]\to[0,1]$ be a continuous function such that $x^2+(f(x))^2\le 1$ for all $x\in[0,1]$ and $\int_0^1 f(x)\,dx=\frac{\pi}{4}$ Then $\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}}\frac{f(x)}{1-x^2}dx$ equals

(a) $\frac{\pi}{12}$ (b) $\frac{\pi}{15}$ (c) $\frac{\sqrt{2}-1}{2}\pi$ (d) $\frac{\pi}{10}$

ANSWER KEY

1. b	**2.** b	**3.** b	**4.** a	**5.** b
6. b	**7.** a	**8.** b	**9.** b	**10.** c
11. c	**12.** c	**13.** a	**14.** c	**15.** a
16. c	**17.** d	**18.** a	**19.** a	**20.** d
21. b	**22.** a	**23.** b	**24.** a	**25.** d
26. a	**27.** d	**28.** a	**29.** b	**30.** a

HINTS & SOLUTIONS

1.Sol: First, we choose one of the three letters to be the letter that is repeated. That's three possibilities.

Next, we find all distinct permutations of our four letters. There is a letter that appears twice, and two other letters that each appear once. To count these permutations, we use the multinomial coefficient $\binom{4}{2,1,1}=\frac{4!}{2!1!1!}=12.$

Therefore, total number four-letter words can be formed with the given conditions is $3\times 12=36$.

2.Sol: Given

$$\left(\frac{1}{3}sin(\theta)+\frac{2}{3}cos(\theta)\right)^2=\frac{1}{3}sin^2(\theta)+\frac{2}{3}cos^2(\theta)$$

Upon simplification, above equation reduces to

$$\frac{1}{9}sin^2(\theta)+\frac{4}{9}cos^2(\theta)+\frac{4}{9}sin(\theta)cos(\theta)$$

$$=\frac{1}{3}sin^2(\theta)+\frac{2}{3}cos^2(\theta)$$

That is $2\sin\theta\cos\theta=sin^2\theta+cos^2\theta$

$$\Rightarrow sin(2\theta)=1$$

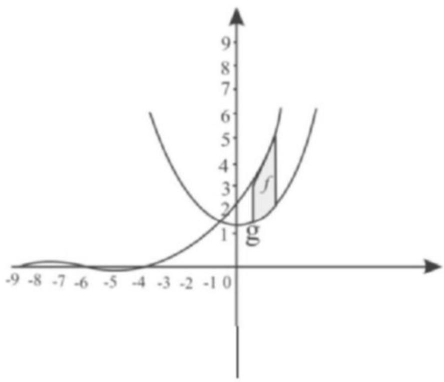
i.e. to $sin(2\theta)=1$, which is possible only for $\theta=\frac{\pi}{4}$ in $[0,\pi]$. So the given equation has only one solution

3.Sol: Since x never vanishes in $[1,2]$, we can divide by x and recast the equations of the two curves as

$$y=f(x)=\frac{\sin x}{x}+e^x \qquad (1)$$

and $$y = g(x) = \frac{\sin x}{x} + \frac{x^2}{2} \quad (2)$$

respectively. For all $x \geq 0, e^x \geq 1 + x + \frac{x^2}{2}$ as can be seen by repeated differentiation. So the graph of $y = f(x)$ is above that of $y = g(x)$ for all $1 \leq x \leq 2$.



Therefore, the area, say A, of the region between them is shown in above figure is given by A

$$AB = \int_1^2 \left(e^x - \frac{x^2}{2}\right) dx$$

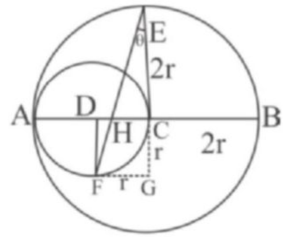
$$= \left[e^x - \frac{x^3}{6}\right]_1^2$$

$$= e^2 - \frac{8}{6} - e + \frac{1}{6}$$

$$= \int_1^2 \left\{\frac{\sin x}{x} + e^x - \left(\frac{\sin x}{x} + \frac{x^2}{2}\right)\right\} dx$$

$$= e^2 - e - \frac{7}{6}.$$

4.Sol: Let $\angle FEC = \theta$. Let FE meet AB at H. It is a little complicated to find $\sin\theta$ from the right angled triangle HCE. But if we draw a line through F which is parallel to AB and let it meet CF (extended) at G, then $\sin\theta$ can be found easily from the right-angled triangle FGE. Let $2r$ and r be the radii of C_1 and C_2 respectively. Then by direct calculation, $FE = \sqrt{r^2 + 9r^2} = \sqrt{10}r$. Since $FG = r$, we have $\sin\theta = \frac{FG}{FE} = \frac{r}{\sqrt{10}r} = \frac{1}{\sqrt{10}}$



5.Sol: The determinant can be simplified by taking out x and x^2 as common factors from the second and the third columns respectively. That gives

$$\det\begin{bmatrix}1 & x & x^2\\ 1 & x^2 & x^4\\ 1 & x^3 & x^6\end{bmatrix} = x^3 \det\begin{bmatrix}1 & 1 & 1\\ 1 & x & x^2\\ 1 & x^2 & x^4\end{bmatrix}$$

The determinant on the R.H.S. is a Vandermonde determinant, which can be expanded as $(x-1)(x^2-1)(x^2-x)$. But here it can as well be expanded directly as $x^5 - x^4 + x^2 - x^4 + x^2 - x = x^5 - 2x^4 + 2x^2 - x$. So the given equation reduces to $x^8 - 2x^7 + 2x^5 - x^4 = 3x^4$, Which has *0* as a root of multiplicity 4. Canceling the factor we are left with

$$x^4 - 2x^3 + 2x - 4 = 0$$

The L.H.S factors as $(x^3 + 2)(x - 2)$. So 2 is also a root of the given equation. The first factor $x^3 + 2$ has only one real root $(-2)^{\frac{1}{3}}$. But it is not an integer. Nor are the other two roots which are complex.

Hence the given equation has only two integral roots, viz. 0 and 2.

6.Sol: $P(x)$ is some polynomial in x. The data implies that the graph of $y=P(x)$ is symmetric about the line $x=1$ and $P(1)=0$. To work with this data effectively, it is convenient to introduce a new polynomial $Q(x)$ defined as $P(1-x)$. This is also a polynomial in x with the same degree as P. (In fact, P and Q determine each other uniquely since $Q(1-x)=P(x)$. But we now have $Q(-x)=P(1+x)=P(1-x)=Q(x)$. Also $Q(0)=P(1)=0$. Hence $Q(x)$ is an even function. So it has only even degree terms and the constant term is 0. Hence $Q(x)$ is always divisible by x^2. Moreover 2 is the largest value of m such that x^m divides all such polynomials $Q(x)$. It can be no higher since x^2 is a polynomials of this type.

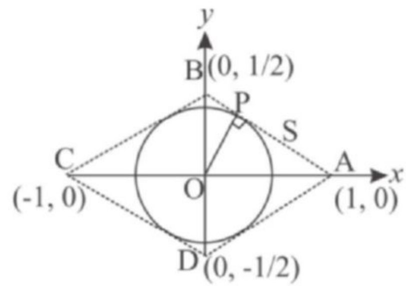
Since x^2 divides $Q(x)$, $(1-u)^2$ divides $Q(1-u)$ which is simply.

$P(1-(1-u))=P(u)$. Hence $(1-u)^2$ divides $P(u)$. Changing the dummy variable from u to x, we get $(1-x)^2$ which also equal $(x-1)^2$ divides $P(x)$.

7.Sol: Since $f(0)$ is explicitly given as $1, 0\in A$ and so A has at least one element. To see if it has any other, first suppose $x>0$. Since $\sin 1\neq 1$ For $0<x<1, |f(x)|=x|\sin 1|\le x\le 1$ (since $1<\frac{\pi}{2}$ and so $x\notin A$. Finally, suppose x > 1. Then $0<\frac{1}{x}<1<\frac{\pi}{2}$ and so by a well known inequality (which says $\sin\theta<\theta$ for an acute angle θ), $\frac{1}{x}>\sin\left(\frac{1}{x}\right)$. Therefore $x\sin\left(\frac{1}{x}\right)<1$ and so $x\notin A$.

Thus we have shown that A has no positive elements. Since $f(x)$ is an even function (i.e. $f(-x)=f(x)$ for all $x\in R$) it follows that A can have no negative elements either. Therefore A has only one element viz. 0.

8.Sol: The set S is the boundary of a rhombus with verticles at $(\pm1,0)$ and $(0,\pm1)$ shown in the figure below.



The centre of the rhombus is at the origin and so the circle in the question is simply the incircle of this rhombus. Its radius is the shortest distance of O From S. By symmetry, we can take the perpendicular distance, shown as OP from any one side of S. As the equation of this side $x+2y=1$, is the perpendicular distance from O is $\frac{1}{\sqrt{1+4}}=\frac{1}{\sqrt{5}}$.

9.Sol: Putting $\theta=3x$ and using the identity $\sin 3\theta=3\sin\theta-4\sin^3\theta$ reduces the given equation to $4\sin\theta-4\sin^3\theta=0$.

Whose possible solutions are $\sin\theta=0$ and $\sin\theta=\pm1$. As x varies from 0 to 2π, θ varies from 0 to 6π. In the interval $[0,6\pi]$, there are 7 zeros of the sine function, viz. all integral multiples of π from 0 to 6π. $\sin\theta=\pm1$ occurs when θ is an odd multiple of $\frac{\pi}{2}$. In the interval $[0,6\pi]$, there are 6 such multiples, from $\frac{\pi}{2}$ to $\frac{11\pi}{2}$. Hence the total number of solutions is $7+6=13$.

10.Sol: The diagonals of any parallelogram bisect each other and these semi-diagonals from four triangles of equal areas. In the present case, these semi-diagonals are 5 and 2. The areas of these four triangles will be maximum when these diagonals are at right angle to each other. In that case, the parallelogram will be a rhombus with each side equal to $\sqrt{5^2+2^2}=\sqrt{29}$ units in length. So the perimeter, say p, will be $4\sqrt{29}$. Since $\sqrt{29}>5,(A)$ is ruled out. For the other options, it is convenient to take

$p^2=16\times 29=464$.

Since $21^2=441, 22^2=484$ we see that $21<P<22$ Hence (C) holds.

11.Sol: Since a,b are positive integers, then $2a-1$ $2b-1$ and are odd, positive integers. For $\frac{2a-1}{b}$ and $\frac{2b-1}{a}$ to both be integers requires that a and b be odd, with the resulting integers from those fractions also being positive and odd. In particular, you have for some odd, positive integers i and j that $\frac{2a-1}{b}=i$, which simplifies to

$$2a=ib+1 \qquad (1)$$

Similarly $\frac{2b-1}{a}=j$, simplifies to

$$2b=ja+1 \qquad (2)$$

Substituting (1) into (2), rearranging and factoring gives

$$\left.\begin{aligned} 4b-2&=j(ib+1),\\ 4b-jib&=j+2\\ (4-ji)b&=j+2 \end{aligned}\right\} \qquad (3)$$

Since b and $j+2$ are both positive, $4-ji$ must also be positive. As i, j are odd, positive integers, the only 3 possibilities for them are $i=j=1, i=1, j=3$ and $i=3, j=1$.

Substituting these into (3) to get b and then substituting the values into (2) to get a gives $(a,b)=(1,1),(a,b)=(3,5)$ and $(a,b)=(5,3)$, respectively. This matches what we already determined to be 3 solutions, but it also shows there are no others.

12.Sol: Let $Z=z^2$ and $W=w^2$ and then take the square roots of Z and W. Note that Z, W also lie on the unit circle since $|Z|=|z|^2=1$ and $|W|=|w|^2=1$. So, if we let $Z=A+iB$ and $W=C+iD$, then $A^2+B^2=C^2+D^2=1$. With this information, the equation

$$Z+W=1$$

means $A+C=1$ and $D=-B$. So $D^2=B^2$ which means $A^2=C^2$. Hence $C=\pm A$. Therefore $A+C=1$ can hold only when $A=C=\frac{1}{2}$. The values of B and D are $\pm\frac{3}{2}$ or vice versa. So there are two possible values Z and each determines W uniquely. But each of Z and W has two distinct square roots. Since the roots of Z and W can be clubbed together independently, in all we have $2\times 2\times 2=8$ possible ordered pairs (z,w) satisfying the given conditions.

13.Sol: Let V denotes vowels and C denotes consonants. Then total number of 4- letter words is $(26)^4$.

Number of 4 - letter words which contains only vowels is $(5)^4$.

Number of 4 - letter words which contains only consonants is $(21)^4$.
Number of words which contains at least on vowel and at least one consonants is

$$(26)^4-(21)^4-5^4=261870.$$

14.Sol: For each of the three planes, we are given a point. viz. $(0,0,0)$, on it and a vector perpendicular to it. This gives the equations of the planes $\sigma_1,\sigma_2,\sigma_3$ is

$$x+y+z=0 \qquad (1)$$

$$ax+by+cz=0 \qquad (2)$$

and $$a^2x+b^2y+c^2z=0 \qquad (3)$$

respectively.
This is a homogeneous system of three linear equations in three unknown *x, y, z*. The problem asks to decide under which of the given conditions it has no solution other than $x=0, y=0, z=0$. The necessary and sufficient condition for this is that the determinant *D* of the coefficients given by

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

is non-zero. That is $(a-b)(b-c)(c-a)\neq 0$.
So, we now have to identify which of the given four options ensures that *D* is non-zero. All the conditions assume that *a, b,* and *c* are positive.

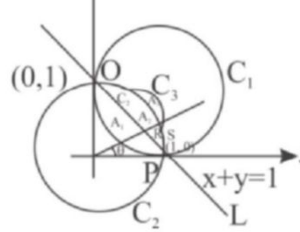
Thus we see that (*C*) maskes *D* non-zero regardless of the signs of *a, b, c*.

15.Sol: We denote a red card by *r* and a black card by *b*. At any stage, it suffices to keep track of the cards held by one of the players, since that determines the cards held by the other. We choose to keep track of ravi's cards. His/her initial holding is (2*r*, 2*b*). After getting a card from Rashmi, it would change either to (3*r*, 2*b*) or to (2*r*,3*b*) depending on the colour of the card picked. Each occurs with probability $\frac{1}{2}$. Assuming that the first possibility holds, the next holding of Ravi will be (*3r,* 2*b*) with probability $\frac{2}{5}$ and (*2r,* 3*b*) with probability $\frac{3}{5}$.

A similar reasoning applies when Ravi holds (*2r,3b*) and Rashmi picks one card at random from these five cards.

Since we are interested only in the cases when Ravi's last holding is either *4r* or *4b,* there is no point thinking about (*2r,2b*) in the second hold . However, from (*3r,b*),*4r* can be reached if Ravi picks a red card from Rashmi,The probabilities of these are $\frac{1}{4}$ and $\frac{1}{5}$ respectively. Hence from(*2r,2b*) at the start, 4*r* can be reached with probability $\frac{1}{5}\times\frac{1}{4}\times\frac{1}{5}=\frac{1}{100}$. By symmetry, 4*b* can also be reached with probability $\frac{1}{100}$. The two probabilities add to 2%, which is less than 5.followed by Rashmi picking the black card from Ravi.Hence (A) holds

16.Sol: All the regions are in the first quadrant. Further, all of them involve the line, say *L*, with equation $x+y=1$. A_2 is the region lying between *L* and the unit circle C_2 while A_3 is the region between *L* and the curve C_3. To identify A_1, we complete the squares and rewrite the first inequality as $(x-1)^2+(y-1)^2<1$. Then A_1 is the region between the line *L* and the circle of radius 1 centred at $(1,1)$. The circles C_1 and C_2 have equal radii and a common



chord. Thus they are the mirror images of each other in the line L. Therefore the regions A_1 and A_2 are congruent to each other and so have equal areas.

To compare $|A_2|$ with $|A_3|$, we claim that $A_2 \subset A_3$. we get $|A_2|<|A_3|$. Hence (C) holds.

17.Sol: Obviously, every constant function satisfies the condition in the question. We claim that the constant functions are the only continuous functions which satisfy $f(x^2) = f(x^3)$ for all $x \in R$. Call x^3 as y. Then $x = y^{\frac{1}{3}}$ and so this equality reads as

$$f\left(y^{\frac{2}{3}}\right) = f(y) \qquad (1)$$

As x varies through all real numbers, so does y. So (1) is valid for all $y \in R$. Applying it again with y replaced by $y^{\frac{2}{3}}$ gives

$$f\left(y^{\frac{4}{9}}\right) = f\left(y^{\frac{2}{3}}\right) \qquad (2)$$

because of the law of indices $\left(y^a\right)^b = y^{ab}$. But now we can apply (1) again with y replaced by $y^{\frac{4}{9}}$ to get $f\left(y^{\frac{8}{27}}\right) = f\left(y^{\frac{4}{9}}\right)$. In general, for every positive integer n we have

$$f\left(y^{\left(\frac{2}{3}\right)^n}\right) = f(x) \qquad (3)$$

As $n \to \infty$, the exponent $\left(\frac{2}{3}\right)^n \to 0$. So for $y \neq 0, y^{\left(\frac{2}{3}\right)^n} \to 1$. Hence by continuity of f, $f\left(y^{\left(\frac{2}{3}\right)^n}\right) \to f(1)$ as $n \to \infty$. Therefore (3) implies that $f(y) = f(1)$ for all $y \neq 0$. By continuity of f, this also holds for $y = 0$. Thus, $f(x)$ is a constant function, the constant being $f(1)$.

Since f is a constant function, II and III hold trivially. So (D) is true.

18.Sol: As $f(t)$ is continuous, so is the function $t\,f(t)$. Therefore the function defined by integrating it from 0 to x is a differentiable function of x. If we differentiate both the sides of the given equation using the second form of fundamental theorem of calculus, we get

$$f'(x) = 2xf(x) \qquad (1)$$

for all $x \geq 0$. Solving this differential equation, we get $\ln(f(x)) = x^2 + c$ for some constant c.

Hence $f(x) = ke^{x^2}$ for some constant k. The given relation implies $f(0) = 1$. Hence $k = 1$ and so $f(x) = e^{x^2}$ for all $x \geq 0$. Hence $f(1) = e^1 = e$.

19.Sol: We rewrite the equation as a quadratic equation in b.

$$\left(1+a^2\right)b^2 - 4ab + \left(1+a^2\right) = 0$$

We have to find the solution set of this quadratic, for a given $a > 0$, $a \neq 1$. Before applying the quadratic formula, we examine the discriminant $D = 4a^2 - \left(1+a^2\right)^2$. Rewriting it as $-\left(1+a^2\right)^2$ we see that it is negative for $a \neq 1$. So the quadratic has no solution in b. Hence the solution set is empty.

Method.2: A slightly easier solution using the A.M.-G.M. inequality is possible if we use a, b are positive and $a \neq 1$.

$$(1+a^2)(1+b^2) = 4ab$$

Implies that

$$\left(\frac{1}{a}+a\right)\left(\frac{1}{b}+b\right) \geq 4$$

Since the given expression in the question is a particular case of the above expression. Therefore the given expression holds when $a = b = 1$. Since $a \neq 1$ so b has no positive real number satisfying the given condition.

20.Sol: Near $x = 0$, sin$x$ is comparable to x in the sense that the ratio $\frac{\sin x}{x}$ tends to 1 as $x \to 0$.

This also holds if we replace x by x^2.

Therefore $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin(x^2)}{x}$

$$= \lim_{x\to 0} \frac{\sin(x^2)}{x^2} x$$

$$= 1 \times 0 = 0 = f(0) \qquad (1)$$

which shows that f is continuous at 0.

As for differentiability of f at $x \neq 0$, we get by the chain rule that $f'(x)$

$$= 2\cos x^2 - \frac{\sin x^2}{x^2} \qquad (2)$$

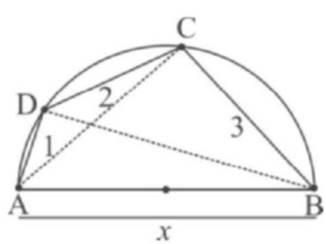
$$f'(x) = \frac{2x^2 \cos(x^2) - \sin(x^2)}{x^2}$$

As $x \to 0$ this tends to $2-1=1$ so $f'(x)$ exist at $x \neq 0$. For differentiability at 0, we have to proceed from the definition. For $x \neq 0$, we have

$$\frac{f(x) - f(0)}{x - 0} = \frac{\sin(x^2)}{x} \qquad (3)$$

which tends to 1 as $x \to 0$. Therefore $f'(x)$ exists and equals 1. As $\lim_{x\to 0} f'(x)$ also equals 1,we get that f is differentiable at 0 and also that its derivative is continuous at 0.

21.Sol: Let x be the diameter. Then $AD = \sqrt{x^2 - 9}$ and $BC = \sqrt{x^2 - 1}$. Using



Ptolemy's theorem which says $AB.CD + AC.BD = AD.BC$.That is

$$x \cdot 2 + 1 \cdot 3 = \sqrt{x^2 - 9}\sqrt{x^2 - 1}.$$

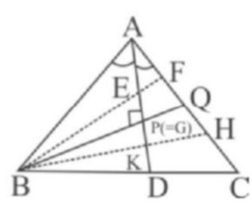
Simplification and cancellation by x gives a cubic equation in x, viz.

$$x^3 - 14x - 12 = 0 \qquad (1)$$

This cubic has no obvious root. The very fact that the question does not ask for the exactvalue of x but only to identify an interval which contains it, suggests that an attempt to find an exact root is not likely to succeed. If we call the L.H.S. of (1) as $f(x)$, then $f(4) = -4 < 0$. But $f(5) = 43 > 0$. So, by the Intermediate Value Property, $f(x)$ vanishes at least onepoint in the open interval $(4,5)$.

To narrow this interval further, we need to compute $f(4.1), f(4.2)$ and $f(4.3)$. By a direct calculation, the first two come out to be -0.479 and 3.288 respectively. As they are of opposite signs, we get that x lies in the interval $(4.1, 4.2)$. So, without computing $f(4.3)$,(B) holds.

22.Sol: Let the perpendicular from B to AD meet it at P and AC at Q .We claim $P = G$ and hence $\angle BGA = 90^\circ$.



Since $\angle CAD = \angle QAP$ and $\angle BAD = \angle BAP$, and the triangles QAP and BAP are right angled at P, $\cot\angle CAD = \frac{AP}{PQ}$ and $\cot\angle BAD = \frac{AP}{BP}$. Therefore the given equality about their ratios implies that $BP = 2PQ$. So the segment BQ is divided in the ratio 2: 1 by the median AD. On the other hand, since G is the centroid, the median through B is also divided in the ratio 2: 1. But there is only one line through B which has this property. For a typical dotted line like BEF shown, the ratio $BE : EF$ decreases from ∞ to 1 as E moves along the median AD from A to D. The centroid is the only position when this ratio is 2: 1. So, P has to equal G. Therefore

$\angle BGA = \angle BPA = 90^\circ$.

23.Sol: Let $S = f(a)+f(b)+f(c)+f(d)$. Express $f(x)$ as

$$f(x) = q(x)g(x) + r(x) \quad (1)$$

where $q(x)$ and $r(x)$ are some polynomials and the degree of $r(x)$ is less than that of $g(x)$. Once we do this we have

$$f(a) = q(a)g(a) + r(a) = r(a) \quad (2)$$

and similarly, $f(b)$, $f(c)$, $f(d)$ equal $r(b), r(c), r(d)$ respectively. That reduces the problem to a similar problem, with $f(x)$ replaced by $r(x)$ which is of degree at most 3 and hence more manageable.

To recast $f(x)$ in the form (1), we carry out the long division. We skip the details but write down the final result which can be verified by a direct multiplication.

$$x^6 - 2x^5 + x^3 + x^2 - x - 1$$

$$= \left(x^2 - x\right)\left(x^4 - x^3 - x^2 - 1\right) + 2x^2 - 2x - 1 \quad (3)$$

Therefore, by what we said above,

$$S = 2\left(a^2 + b^2 + c^2 + d^2\right) - 2\left(a + b + c + d\right) - 4$$

Factoring $g(x) = x^4 - x^3 - x^2 - 1$ as

$(x-a)(x-b)(x-c)(x-d)$ and equating the coefficients of the like powers, we get four well-known formulas of which we need only two, viz.

$$a + b + c + d = 1 \quad (5)$$

and

$$ab + ac + ad + bd + bc + cd = -1 \quad (6)$$

To use (4), we need $a^2 + b^2 + c^2 + d^2$ (or s^2 in our earlier notation). That can be found from (5) and (6) as

$$a^2 + b^2 + c^2 + d^2 = \left(a + b + c + d\right)^2$$

$$-2\left(ab + ac + ad + bd + bc + cd\right)$$

$$1 + 2 = 3$$

A straight substitution into (4) gives $S = 0$.

24.Sol: (i) $\vec{r} \cdot \vec{a} = 5$

This is an equation of plane

(ii) $\left|\vec{r} - \vec{b}\right| + \left|\vec{r} - \vec{c}\right| = 4$

i.e. sum of distances of a point $\left(\vec{r}\right)$ from two fixed points with position vector $\vec{b}$ and $\vec{c}$ is constant (Also check $\left|\vec{b} - \vec{c}\right| = \sqrt{14} < 4$)

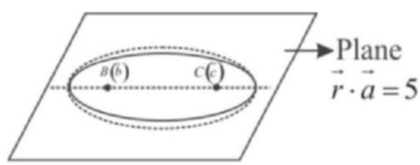
$\Rightarrow$ such points lies on ellipsoid (as in 2D, such points lies on ellipse)

Now points with p.v. $\vec{b}$ and $\vec{c}$ satisfies the equation of plane

$$\vec{b} \cdot \vec{a} = 5$$

$$\vec{c} \cdot \vec{a} = 5$$ CJSFH

DI



Area in the plane constitutes an ellipse

Distance between $\vec{b}$ and $\vec{c} = 2 \times$ (semi major axis) $\times e = \sqrt{14}$

i.e. $2ae = \sqrt{14}$ (1)

Sum of distance is constant that is major axis $= 4$ and $2a = 4$ (2)

$$\Rightarrow e = \frac{\sqrt{14}}{4} \Rightarrow b = \frac{1}{\sqrt{2}} \text{ (semi minor axis)}$$

Area of ellipse $= \pi \cdot a \cdot b$

$$= \pi \cdot 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}\pi \approx 4.443$$

25.Sol: As $\pi\sin^2\theta$ and $\pi\cos\theta$ add to π, the terms on the L.H.S. are equal. So the equation can be recast as $(\pi\cos^2\theta)=\cos\left(\frac{\pi}{2}\cos\theta\right)$ and further as

$$\cos\left(\frac{\pi}{2}-\pi\cos^2(\theta)\right)=\cos\left(\frac{\pi}{2}\cos\theta\right) \quad (1)$$

which has a general solution of the form

$$\frac{\pi}{2}-\pi\cos^2(\theta)=2n\pi\pm\frac{\pi}{2}\cos\theta \quad (2)$$

Where n is an integer. Simplifying,

$$2\cos^2\theta\pm\cos\theta+4n-1=0 \quad (3)$$

This is a quadratic in $\cos\theta$, with solutions

$$\cos\theta=\frac{\pm1\pm\sqrt{9-32n}}{4} \quad (4)$$

Positive values of the integer n are inadmissible. Also if n is a negative integer then the radical is at least $\sqrt{41}$ and so the values of $\cos\theta$ are numercially greater than 1. So $n=0$ is the only possibility to consider. For $n=0$, the possible values of $\cos\theta$ are ±1 and $\pm\frac{1}{2}$. The first two possibilities imply $\theta=0,\pi$ or 2π. Each one of these is a solution. For $\cos\theta=\pm\frac{1}{2}$ there are two possible values each in $[0,2\pi]$. Hence in all the equation has 7 solutions in $[0,2\pi]$.

26.Sol: The substitution $x^2=t$ converts J to the integral $\frac{1}{2}\int_0^1\frac{dt}{1+t^4}$. For every $t\in[0,1], 1+t^4 \le 1+t^2$ and so $\frac{1}{1+t^4}\ge\frac{1}{1+t^2}$.

Therefore,

$$J=\frac{1}{2}\int_0^1\frac{dt}{1+t^4}\ge\frac{1}{2}\int_0^1 1dt$$

$$=\frac{1}{2}>\frac{1}{4} \quad (2)$$

Hence I is true. So, only I is true.

27.Sol: Let $y=f(x)$. Then y satisfies the differential equation

$$(y')^4=16y^2 \quad (1)$$

with the condition $y(0)=0$. Often the point at which the solution is stipulated to have a given is an initial point of some interval.

We can recast the equation to a collection of several first order equations by taking the fourth roots of both the sides to get

$$y'=\pm2\sqrt{y}=\pm2y^{\frac{1}{2}} \quad (2)$$

or, after splitting the variables, as

$$\frac{1}{2}y^{-\frac{1}{2}}dy=\pm1dx \quad (3)$$

As straight integration gives the general solution as

$$\sqrt{y}=\pm x+c \quad (4)$$

Where c is some constant. The condition $y=0$ when $x=0$ determines $c=0$. Hence the solution is $y=x$. In addition, $y=-x$ is also a solution as can be verified by a direct calculation.

So, we now have three solutions:

$(i)\,y=x^2, (ii)\,y=-x^2$ and $(iii)\,y=0$ for $x\in(-1,1)$ which all satisfy the condition $f(0)=0$, i.e. $y=0$. Here 0 is an interior and not an end point of the domain interval $(-1,1)$. So, we are effectively dealing with two IVP's one on $[0,1)$ and the other on $(-1,0]$. On each one of these two we have a team of three solutions. But we can match a solution of one team with any of the three members of the other team.

Since there are three possible choices in each half, in all the differential equation has 9 possible solutions satisfying the given condition. So (*d*) is correct.

28.Sol: Given $f(x)=|\sin x|$, $g(x)=\int_0^x f(t).dt$

and $\quad p(x)=g(x)-\frac{2}{\pi}x$

Now, $p(x+\pi)=g(x+\pi)-\frac{2}{\pi}(x+\pi)$

$$=\int_0^{\pi+x} f(t)dt.-\frac{2}{\pi}x-2$$

$$=\int_0^{x} f(t)dt+\int_0^{\pi+x} f(t)dt.-\frac{2}{\pi}x-2$$

[$f(x)$ is periodic function with period π,

therefore, $\int_0^{\pi+x} f(t)dt=\int_0^{x} f(t)dt$]

$$\Rightarrow p(x+\pi)=\int_0^{x}|\sin x|dx+g(x)-\frac{2}{\pi}x-2$$

$$=2+g(x)-\frac{2}{\pi}x-2$$

$p(x+\pi)=p(x)$ for all x

29.Sol: Given $\left(\sum_{i=1}^{3}\frac{a_i}{2^i}\right)^2=\sum_{i=1}^{3}\frac{a_i^2}{2^i}$

$$\left(\frac{a_1}{2}+\frac{a_2}{4}+\frac{a_3}{8}\right)^2=\frac{a_1^2}{2}+\frac{a_2^2}{4}+\frac{a_3^2}{8}$$

Simplifying, we get

$$\frac{a_1^2}{4}+\frac{a_2^2}{16}+\frac{a_3^2}{64}+\frac{a_1a_2}{4}+\frac{a_1a_3}{8}+\frac{a_2a_3}{16}=\frac{a_1^2}{2}+\frac{a_2^2}{4}+\frac{a_3^2}{8}$$

$$16a_1^2+12a_2^2+7a_3^2=16a_1a_2+8a_1a_3+4a_2a_3$$

$$\left(8a_1^2+8a_2^2-16a_1a_2\right)+\left(8a_1^2+2a_3^2-8a_1a_3\right)$$

$$+\left(4a_2^2+a_3^2-4a_2a_3\right)+4a_3^2=0$$

$$8\left(a_1-a_2\right)^2+2\left(2a_1-a_3\right)^2+\left(2a_2-a_3\right)^2+4a_3^2=0$$

$$\left.\begin{array}{l}a_1-a_2=0\\2a_1-a_3=0\\\Rightarrow 2a_2-a_3=0\\a_3=0\end{array}\right\}\Rightarrow a_1=a_2=a_3=0$$

A contains exactly one element

30.Sol: The given inequality on $f(x)$, viz,

$$x^2+\left(f(x)\right)^2\le 1 \qquad (1)$$

implies, first of all, that

$$f(x)\le\sqrt{1-x^2} \qquad (2)$$

for all $x\in[0,1]$. Therefore,

$$\int_0^1 f(x)dx\le\int_0^1\sqrt{1-x^2}dx \qquad (3)$$

The last integral is π as we see by putting $x=sin\,\theta$.

So, even though (2) is only an inequality about two functions, the integrals of both the functions are the same. From this and continuity of $f(x)$, we claim that equality must hold throughout in (2). So we must have $f(x)=\sqrt{1-x^2}$ for all $x\in[0,1]$. It is now easy to finish the solution , because

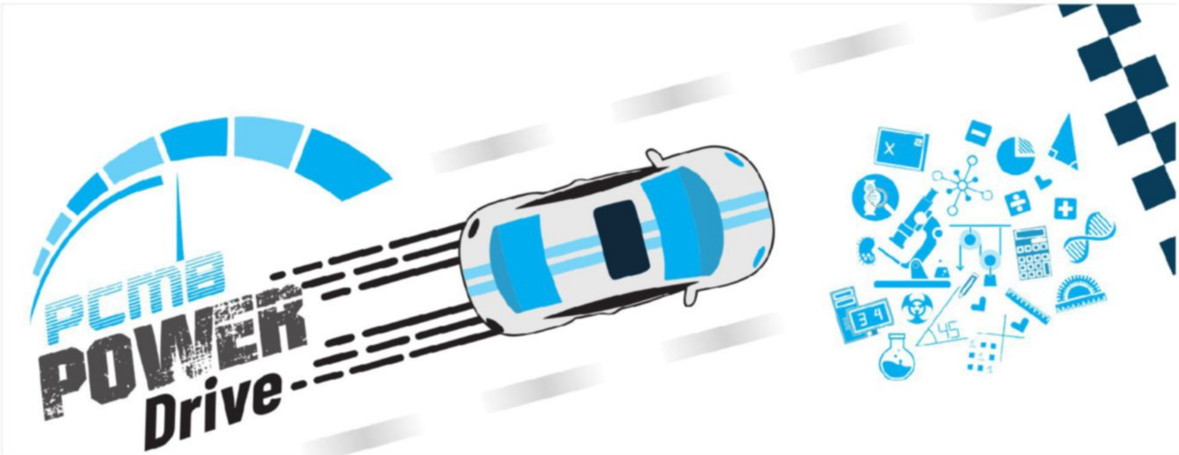
$$\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}}\frac{f(x)}{\sqrt{1-x^2}}dx=\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}}\frac{dx}{\sqrt{1-x^2}}$$

$$=\sin^{-1}x\Big|_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}}$$

$$=sin^{-1}\left(\frac{1}{\sqrt{2}}\right)-sin^{-1}\left(\frac{1}{2}\right)$$

$$=\frac{\pi}{4}-\frac{\pi}{6}=\frac{\pi}{12}$$

So (A) is true.



CONTINUITY & DIFFERENTIABILITY

(1) Continuity

Definition: A function f is continuous at $x = a$

(i) $f(a)$ is defined

(ii) $\lim_{x \to a} f(x)$ exists, and

(iii) $\lim_{x \to a} f(x) = f(a)$.

The definition of continuity all boils down to the one condition in (iii), since conditions (i) and (ii) must hold whenever (iii) is meet. Further, this says that a function is continuous at a point exactly when you can compute its limit at that point by simply substituting in.

(2) Discontinuity of Functions

A function $f(x)$, which is not continuous at a point $x = a$, is said to be discontinuous at that point. The discontinuity may arise due to any of the following reasons

- $\lim_{h \to 0} f(a-h) \neq \lim_{h \to 0} f(a+h)$, i.e., LHL and RHL exist, but are not equal
- $\lim_{h \to 0} f(a-h) = \lim_{h \to 0} f(a+h) \neq f(a)$, LHL and RHL exist and are equal, but are different from $f(a)$.
- $f(a)$ is not defined.
- Atleast one of the limits $\lim_{h \to 0} f(a-h)$ or $\lim_{h \to 0} f(a+h)$ does not exist or atleast one of these limits is ∞ or $-\infty$.

(3) Free Theorems

Theorem:

(1) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(2) Any rational function is continuous where ever it is defined; that is, it is continuous in its domain.

(4) The Limit Law gives Continuity Law

Theorem: Suppose that f and g are continuous at $x = a$. Then all of the following are true

(1) $(f \pm g)$ is continuous at $x = a$

(2) $(f \cdot g)$ is continuous at $x = a$ and

(3) $\left(\dfrac{f}{g}\right)$ is continuous at $x = a$ if $g(a) \neq 0$.

(5) Transcendental Functions Theorem

The following functions are continuous wherever they are defined:

(1) $\sin(x), \cos(x), \tan(x), \cot(x), \sec(x), \csc(x)$.

(2) $a^x, \log_a x, \ln x, e^x$

(3) $\sin^{-1}(x), \cos^{-1}(x), \tan^{-1}(x)$

(6) Continuity of Composite Functions

Theorem: Suppose that $\lim_{x \to a} g(x) = L$ and f is continuous at L. Then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L).$$

Notice that this says that if f is continuous, we can bring the limit "inside." This should make sense, since as $x \to a$, $g(x) \to L$ and so, $f(g(x)) \to f(L)$, Since f is continuous at L.

(7) Corollary : Suppose that g is continuous at a and f is continuous at $g(a)$. Then, the composition of fog is continuous at a.

Definition :- If f is continuous at every point on an open interval (a,b), we say that f is continuous on (a,b).

Definition :- If f is continuous on the closed interval $[a,b]$, if f is continuous on the open interval (a,b) and

$$\lim_{x\to a+} f(x) = f(a) \text{ and } \lim_{x\to b-} f(x) = f(b).$$

(8) Intermediate Value Theorem

Suppose $f(x)$ is continuous on the closed interval $[a,b]$ and y is any number between $f(a)$ and $f(b)$. Then, there is a number $x \in [a,b]$ for which $f(x) = y$.

Consequences:

(I)Intermediate Value Theorem has two important corollaries:

Cor.1: If a continuous function has values of opposite sign inside an interval, then it has a root in that interval (Bolzano's Theorem).

Cor.2: The image of a continuous function over an interval is itself an interval.

(II) Changing signs

The location of roots theorem is one of the most intuitively obvious properties of continuous functions, as it states that if a continuous function attains positive and negative values, it must have a root (i.e. it must pass through 0).

(III) Bolzano's Theorem (BT)

A function can change sign only at roots and Discontinuities. A function $f(x)$ can only change sign (from positive to negative, or vice versa) at a point $x = c$ if the function is zero, undefined, or discontinuous at $x = c$.

(9) Extreme Value Theorem (EVT)

Theorem(EVT): If $f(x)$ is continuous on a closed interval $[a,b]$, then there exist x-values M and m in $[a,b]$ such that $f(M)$ is the maximum value and $f(m)$ is the minimum value of f on $[a,b]$.

(10) Functions to which the theorem does not Apply

Each fails to attain a maximum on the given interval.

(I) $f(x) = x$ defined over $[0,\infty)$ is not bounded from above.

(II) $f(x) = \dfrac{x}{(1+x)}$ defined over $[0,\infty)$ is bounded but does not attain its least upper bound 1.

(III) $f(x) = \dfrac{1}{x}$ defined over $(0,1]$ is not bounded but never attains its least upper bound 1.

(IV) $f(x) = 1 - x$ defined over $(0,1]$ is bounded but never attains its least upper bound 1.

Defining $f(0) = 0$ in the third example shows that both theorems require continuity on $[a,b]$.

(11) Differentiability as a Limit

using one-sided limits, we define the right- hand derivative (RHD), defined by $f'(a^+)$, at $x = a$ as $\lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ and the left hand derivative (LHD) denoted by

$f'(a^-)$, at $x = a$ as

$$\lim_{h\to 0} \frac{f(a-h) - f(a)}{-h}$$

Thus, a function $f(x)$ is derivable at $x = a$ if

$$f'(a^+) = f'(a^-)$$

(12) Differentiability on Domain

Definition: A function is differentiable in D if the function is differentiable at all $x \in D$.

Theorem: If a function is differentiable at a

point c, then it is also continuous at that point.
Corollory: Every differentiable function is Continuous

(13) Non-Differentiable Function Theroem

A function fails to be differentiable at $x = a$ if
(1) Both RHD and LHD exists, finite but are not equal.
(2) Neither LHD nor RHD are not finite.
(3) Neither LHD nor RHD does not exist.

(14) Differentiability at End Points of the Domain Definition

If $f:[a,b] \to R$ then

$$f'(a) = RHD = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$ and

$$f'(b) = LHD = \lim_{h\to 0} \frac{f(b-h)-f(b)}{-h}$$

(15) Differentiability using First Principle

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Remember before proceding to check for differentiability, we must first verify that the function is continuous at 'a' or not.

(16) Differentiability using Graphs

If provided we can draw the graphs of functions and look for the following information.

- If the graph of a function 'f' is discontinuous then it is not differentiable.
- If the graph of a function f is continuous, and but it has sharp edges (tooths), corners then it is not differentiable.

(17) Differentiability using Differentiation

Theorem (chain rule): Let f be a real valued function which is composite of two functions $g(x)$ and $h(x)$; i.e., $f = hog$ suppose

$t = g(x)$ and if both $\frac{dt}{dx}$ and $\frac{d[h(t)]}{dt}$ exist,

we have $\frac{df}{dx} = \frac{d[h(t)]}{dt} \times \frac{dt}{dx}$

(18) Derivative of Implicit Functions

To find $\frac{dy}{dx}$ when a differentiable function $y = f(x)$ satisfies the equation $F(x,y) = 0$, we differentiable F with respect to x, considering y as a function of x.

(19) Derivative of Inverse Functions

Function	Derivative	Remarks
$\frac{d}{dx}\cos^{-1}u$	$\begin{cases} -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \lvert u\rvert<1 \\ +\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \lvert u\rvert<1 \end{cases}$	First and second quadrant Third and fourth quadrant
$\frac{d}{dx}\tan^{-1}u$	$\frac{1}{1+u^2}\frac{du}{dx}$	—
$\frac{d}{dx}\cot^{-1}u$	$-\frac{1}{1+u^2}\frac{du}{dx}$	—
$\frac{d}{dx}\sec^{-1}u$	$\begin{cases} \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}, \lvert u\rvert>1 \\ -\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}, \lvert u\rvert>1 \end{cases}$	First and third quadrant Second and fourth quadrant
$\frac{d}{dx}\mathrm{cos}ec^{-1}u$	$\begin{cases} \frac{-1}{u\sqrt{u^2-1}}\frac{du}{dx}, \lvert u\rvert>1 \\ \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}, \lvert u\rvert>1 \end{cases}$	First and third quadrant Second and fourth quadrant

(20) Derivatives of Logarithmic Functions

Assuming that u is differentiable, we shall obtain the derivative of $\log_a^u$ with respect to x.

i.e., $$\frac{dy}{dx} = \frac{1}{u}\log_a^e \frac{du}{dx} \quad (1)$$

In case $a = e$, the above derivative becomes

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx} \quad (2)$$

(21) Mean Value Theorem

If a function f is continuous on the closed interval $[a,b]$, where $a < b$, and differentiable on the open interval (a,b) then there exists a point c in (a,b) such that:

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

(22) Lagrange's Mean Value Theorem

Theorem: Let $f:[a,b]\to\mathbb{R}$ be a continuous function, differentiable on the open interval (a,b). Then there is atleast one number $c\in(a,b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

(23) Rolle's Theorem

Theorem: Let f be continuous on the closed interval [a,b] and differentiable on the open interval (a,b). If $f(a)=f(b)$, then there is atleast one number c in (a,b) such that $f'(c)=0$.

(24) Cauchy's Mean Value Theorem

Let f and g both be continuous on [a,b] and differentiable on (a,b). Suppose that $g(b)-g(a)\neq 0$ and that f' and g' do not vanish simultaneously. Then there exists a $c\in(a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$$

Exercise

1. If $|f(x)|$ is continuous at $x=a$, then $f(x)$ is

(1) Continuous at $x=a$
(2) Continuous at $x=-a$
(3) Continuous at $x=\sqrt{a}$
(4) Not be continuous at $x=a$

2. $\lim\limits_{x\to 0}\left[(1+3x)^{1/x}\right]=k$, then for continuity at $x=0,k$ is

(1) 3 (2) -3 (3) e^3 (4) e^{-3}

3. If Rolle's theorem for $f(x)=e^x(\sin x-\cos x)$ is verified on $\left[\frac{\pi}{4},\frac{5\pi}{4}\right]$, then the value of c is

(1) $\frac{\pi}{3}$ (2) $\frac{\pi}{2}$ (3) $\frac{3\pi}{4}$ (4) π

4. If f is a continuous function, then

(1) $\int_{-2}^{2}f(x)dx=\int_{0}^{2}\left[f(x)-f(-x)\right]dx$

(2) $\int_{-3}^{5}2f(x)dx=\int_{-6}^{10}f(x-1)dx$

(3) $\int_{-3}^{5}f(x)dx=\int_{-4}^{4}f(x-1)dx$

(4) $\int_{-3}^{5}f(x)dx=\int_{-2}^{6}f(x-1)dx$

5. If the function $f(x)=\begin{cases}(\cos x)^{1/x}, & x\neq 0\\ k, & x=0\end{cases}$ is continuous at $x=0$, then the value of k is

(1) 8 (2) 1
(3) -1 (4) None of these

6. If the function $f(x)=\left[\tan\left(\frac{\pi}{4}+x\right)\right]^{\frac{1}{x}}$ for $x\neq 0$ is $=k$ for $x=0$

(1) e (2) e^{-1} (3) e^2 (4) e^{-2}

7. If $f(x)=\frac{e^{1/x}-1}{e^{1/x}+1},x\neq 0$ and $f(0)=0$, then $f(x)$ is

(1) Left continuous at 0
(2) Right continuous at 0
(3) Discontinuous at 0
(4) Continuous at 0

8. If $f(x)=\begin{cases}\log\left(\sec^2 x\right)^{\cot^2 x} & \text{for } x\neq 0\\ k & \text{for } x=0,\end{cases}$ is continuous at $x=0$, then $k=$

(1) e^{-1} (2) 1 (3) e (4) 0

9. If $f(x)=\begin{cases}ax^2+b, & b\neq 0, x\leq 1\\ bx^2+ax+c, & x>1\end{cases}$, then

$f(x)$ is continuous and differentiable at $x=1$, if

(1) $c=0, a=2b$ (2) $a=b, c\in R$

(3) $a=b, c=0$ (4) $a=b, c\neq 0$

10. The function $f(x)=e^{-|x|}$ is

(1) Continuous everywhere but not differentiable at $x=0$

(2) Continuous and differentiable everywhere

(3) Not Continuous at $x=0$

(4) None of these

11. If [] denotes the greatest integer function, then

$$f(x)=[x]+\left[x+\frac{1}{2}\right]$$

(1) Is continuous at $x=\frac{1}{2}$

(2) Is discontinuous at $x=\frac{1}{2}$

(3) $\lim_{x\to(1/2)^+} f(x)=2$

(4) $\lim_{x\to(1/2)^-} f(x)=1$

12. The set of points where the function $f(x)=|x-1|e^x$ is differentiable, is

(1) R (2) $R-\{1\}$

(3) $R-\{-1\}$ (4) $R-\{0\}$

13. The function $f(x)=[x]\cos\left[\frac{2x-1}{2}\right]\pi$, where $[\cdot]$ denotes the greatest integer function, is discontinuous at

(1) All x

(2) No x

(3) All integer points

(4) x which is not an integer

14. The points of discontinuities of $f(x)=\tan\left(\frac{\pi x}{x+1}\right)$ other than $x=-1$ are

(1) $x=\frac{2m-1}{2m+1}, m\in Z$ (2) $x=\frac{2m+1}{1-2m}, m\in Z$

(3) $x=\pi, 2\pi$ (4) $x=0,\pi$

15. If $f: R\to R$ is defined by

$$f(x)=\begin{cases}\dfrac{2\sin x-\sin 2x}{2x\cos x}, & \text{if } x\neq 0\\ a, & \text{if } x=0\end{cases},$$

then the value of a so that f is continuous at 0 is

(1) 2 (2) 1 (3) -1 (4) 0

16. The function

$$f(x)=\left(\frac{\log_e(1+ax)-\log_e(1-bx)}{x}\right)$$

is undefined at $x=0$. The value which should be assigned to f at $x=0$ so that it is continuous at $x=0$ is

(1) $\frac{a+b}{2}$ (2) $a+b$

(3) $\log(ab)$ (4) $a-b$

17. The function $f(x)=(x^2-1)|x^2-3x+2|+\cos|x|$ is non-differentiable at

(1) -1 (2) 0 (3) 1 (4) 2

18. Let $f(x)$ and $g(x)$ be differentiable function on [0, 2] such that $f''(x)-g''(x)=0$, $f'(1)=2g'(1)=4, f(2)=3g(2)=9$. Then $f(x)-g(x)$ at $x=3/2$ is

(1) 0 (2) 2 (3) 10 (4) 5

19. Let $h(x)=\min(x,x^2)$, for every real numbers x. Then,

(1) h is not continuous for all x

(2) h is differentiable for all x

(3) $h'(x)\neq 1$, for all $x>1$

(4) h is not differentiable at two values of x

20. The function $f(x)=\begin{cases} x^2 & \text{for } x<1 \\ 2-x & \text{for } x\geq 1 \end{cases}$ is

(1) Not differentiable at $x=1$
(2) differentiable at $x=1$
(3) Not continuous at $x=1$
(4) None of these

21. Let $f(x)=\begin{cases} \cos x, & \text{if } x\geq 0 \\ -\cos x, & \text{if } x<0 \end{cases}$

Which one of the following statements is not true ?

(1) $f(x)$ is continuous at $x=1$
(2) $f(x)$ is continuous at $x=-1$
(3) $f(x)$ is continuous at $x=2$
(4) $f(x)$ is continuous at $x=0$

22. If $f(x)=\min\{1,x^2,x^3\}$, then

(1) $f(x)$ is not everywhere continuous
(2) $f(x)$ is continuous and differentiable everywhere
(3) $f(x)$ is not differentiable at two points
(4) $f(x)$ is not differentiable at one point

23. If $f(x)=\dfrac{\log_e(1+x^2\tan x)}{\sin x^3}, x\neq 0,$ is to be continuous at $x=0,$ then $f(0)$ must be defined as

(1) 1 (2) 0 (3) 1/2 (4) -1

24. The number of points at which the function $f(x)=\dfrac{1}{\log_e|x|}$ is discontinuous, is

(1) 1 (2) 2 (3) 3 (4) 4

25. The value f at $x=0$ so that function $f(x)=\dfrac{2^x-2^{-x}}{x}, x\neq 0,$ is continuous at $x=0,$ is :

(1) 0 (2) log 2 (3) 4 (4) log 4

26. Let f and g be differentiable functions such that $f(3)=5, g(3)=7, f'(3)=13,$ $g'(3)=6, f'(7)=2$ and $g'(7)=0.$ If $h(x)=(fog)(x),$ then $h'(3)=$

(1) 14 (2) 12 (3) 16 (4) 0

27. Let R be the set of all real numbers. If, $f:R\to R$ be a function such that $|f(x)-f(y)|^2\leq|x-y|^3, \forall x,y\in R,$ then $f'(x)$ is equal to

(1) $f(x)$ (2) 1
(3) 0 (4) x^2

28. If $g(x)$ is the inverse of $f(x)$ and $f'(x)=\dfrac{1}{1+x^3}$, then $g'(x)$ is equal to

(1) $g(x)$ (2) $1+g(x)$
(3) $1+\{g(x)\}^3$ (4) $\dfrac{1}{1+\{g(x)\}^3}$

29. A differentiable function $f(x)$ is defined for all $x>0$ and satisfies $f(x^3)=4x^4$ for all $x>0.$ The value of $f'(8)$ is :

(1) $\dfrac{16}{3}$ (2) $\dfrac{32}{3}$
(3) $\dfrac{16\sqrt{2}}{3}$ (4) $\dfrac{32\sqrt{2}}{3}$

30. If $f(9)=9, f'(9)=0,$ then $\lim\limits_{x\to 9}\dfrac{\sqrt{f(x)}-3}{\sqrt{x}-3}$ is equal to

(1) 0 (2) $f(0)$ (3) $f'(3)$ (4) $f(9)$

31. Let $f(x+y)=f(x)f(y)$ and $f(x)=1+\sin(3x)g(x),$ where g is

differentiable. The $f'(x)$ is equal to

(1) $3f(x)$ (2) $g(0)$

(3) $f(x)g(0)$ (4) $3g(x)$

32. If $2^x + 2^y = 2^{x+y}$, then the value of $\frac{dy}{dx}$ at (1, 1) is equal to

(1) -2 (2) -1 (3) 0 (4) 1

33. Let $f_n(x)$ be the n- th derivation of $f(x)$. The least value of n so that $f_n = f_{n+1}$, where $f(x) = x^2 + e^x$

(1) 4 (2) 5 (3) 2 (4) 3

34. The functions f, g and h satisfy the relations $f'(x) = g(x+1)$ and $g'(x) = h(x-1)$. Then $f''(2x)$ is equal to

(1) $h(2x)$ (2) $4h(2x)$

(3) $h(2x-1)$ (4) $h(2x+1)$

35. If $y = \frac{x}{x+1} + \frac{x+1}{x}$, then $\frac{d^2y}{dx^2}$ at $x=1$ is equal to

(1) $\frac{-7}{4}$ (2) $\frac{1}{2}$ (3) $-\frac{1}{2}$ (4) $\frac{7}{4}$

36. If $x = a\cos^3\theta$ and $y = a\sin^3\theta$, then $1 + \left(\frac{dy}{dx}\right)^2$ is

(1) $\tan\theta$ (2) $\tan^2\theta$

(3) 1 (4) $\sec^2\theta$

37. If $y = \left(\sin^{-1}x\right)^2$, then $\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx}$ is equal to

(1) 2 (2) -1 (3) -2 (4) 1

38. If $x = a(1+\cos\theta), y = a(\theta + \sin\theta)$, then $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{2}$ is

(1) $-\frac{1}{a}$ (2) $\frac{1}{a}$ (3) $-\frac{1}{2}$ (4) -2

39. If $y = f(x)$ is continuous on [0, 6], differentiable on (0, 6), $f(0) = -2$ and $f(6) = 16$, that at some point between $x=0$ and $x=6, f'(x)$ must be equal to

(1) -18 (2) -3 (3) 3 (4) 14

40. The function $f(x) = \begin{cases} 2x^2 - 1, & \text{if } 1 \le x \le 4 \\ 151 - 30x, & \text{if } 4 < x \le 5 \end{cases}$ is not suitable to apply Rolle's theorem, since

(1) $f(x)$ is not continuous on [1, 5]

(2) $f(1) \ne f(5)$

(3) $f(x)$ is continuous only at $x=4$

(4) $f(x)$ is not differentiable in $x=4$

41. Let $f(x) = x^3 - x + p\ (0 \le x \le 2)$, where p is a constant. The value c of mean value theorem is

(1) $\frac{\sqrt{3}}{2}$ (2) $\frac{\sqrt{6}}{2}$ (3) $\frac{\sqrt{3}}{3}$ (4) $\frac{2\sqrt{3}}{3}$

42. Let f be continuous on [1, 5] and differentiable in (1, 5). If $f(1) = -3$ and $f'(x) \ge 9$ for all $x \in (1,5)$, then :

(1) $f(5) \ge 33$ (2) $f(5) \ge 36$

(3) $f(5) \le 36$ (4) $f(5) \ge 9$

43. The number of discontinuities of the greatest integer function $f(x) = [x], x \in \left(-\frac{7}{2}, 100\right)$ is equal to

(1) 104 (2) 103 (3) 102 (4) 101

44. If $f(x) = \frac{[x]}{|x|}, x \ne 0$ where [.] denotes the greatest integer function, then $f'(1)$ is

(1) -1 (2) ∞
(3) Does not exist (4) None of these

45. If $f(x)=e^x.g(x), g(0)=4$ and $g'(0)=2$, then $f'(0)=$

(1) 1 (2) 3 (3) 2 (4) 6

ANSWER KEY

1. 4	**2.** 3	**3.** 2	**4.** 2	**5.** 2
6. 3	**7.** 3	**8.** 2	**9.** 1	**10.** 1
11. 2	**12.** 2	**13.** 3	**14.** 2	**15.** 4
16. 2	**17.** 4	**18.** 4	**19.** 4	**20.** 1
21. 4	**22.** 4	**23.** 1	**24.** 3	**25.** 4
26. 2	**27.** 3	**28.** 3	**29.** 2	**30.** 1
31. 3	**32.** 2	**33.** 4	**34.** 1	**35.** 4
36. 4	**37.** 1	**38.** 1	**39.** 3	**40.** 4
41. 4	**42.** 1	**43.** 2	**44.** 3	**45.** 4

HINTS & SOLUTIONS

1.Sol: Since $|f(x)|$ is continuous at $x=a$.

$$\therefore \quad |f(x)|=\begin{cases} f(x), & x>a \\ -f(x), & x<a \end{cases}$$

So, $f(x)$ is not continuous at $x=a$.

2.Sol: Given, $\lim_{x\to 0}(1+3x)^{1/x}=k$

Now, $\lim_{x\to 0}(1+3x)^{1/x}=e^3$

$$\therefore \quad k=e^3$$

3.Sol: Given, $f(x)=e^x(\sin x-\cos x)$

On differentiating both sides w.r.t. x, we get

$$f'(x)=e^x\frac{d}{dx}(\sin x-\cos x)$$

$$+(\sin x-\cos x)\frac{d}{dx}(e^x)$$

$$=e^x(\cos x+\sin x)+(\sin x-\cos x)e^x$$

$$=2e^x\sin x$$

We know that, if Rolle's theorem is verified, then their exist $c\in\left(\frac{\pi}{4},\frac{5\pi}{4}\right)$, such that

$$f'(c)=0$$

$$\therefore \quad 2e^c\sin c=0$$

$$\Rightarrow \quad \sin c=0$$

$$\Rightarrow \quad c=\frac{\pi}{2}\in\left(\frac{\pi}{4},\frac{5\pi}{4}\right)$$

4.Sol: Given, $f(x)$ is a continuous function.

Let us consider $f(x)=x$

$$\therefore \quad \int_{-3}^{5}2x\,dx=16$$

$$\text{and} \quad \int_{-6}^{10}(x-1)\,dx=16$$

$$\therefore \quad \int_{-3}^{5}2f(x)\,dx=\int_{-6}^{10}f(x-1)\,dx$$

5.Sol: For $f(x)$ to be continuous at $x=0$, we must have

$$\lim_{x\to 0}f(x)=f(0)$$

$$\Rightarrow \quad \lim_{x\to 0}(\cos x)^{1/x}=k$$

$$\Rightarrow \quad \lim_{x\to 0}\{1+\cos x-1\}^{1/x}=k$$

$$\Rightarrow \quad e^{\lim_{x\to 0}\frac{\cos x-1}{x}}=k$$

$$\Rightarrow \quad e^{-\lim_{x\to 0}\frac{2\sin^2 x/2}{x}}=k$$

$$\Rightarrow \quad e^0=k$$

$$\therefore \quad k=1$$

6.Sol: We have, $f(x)=\left[\tan\left(\frac{\pi}{4}+x\right)\right]^{\frac{1}{x}}=k$

Since, $f(x)$ is continuous at $x=0$, then

$$f(0)=\lim_{x\to 0}f(x)$$

$$=\lim_{x\to 0}\left[\tan\left(\frac{\pi}{4}+x\right)\right]^{1/x}$$

$$\Rightarrow \qquad k = \lim_{x\to 0}\left[\frac{1+\tan x}{1-\tan x}\right]^{1/x} \quad [1^{\infty} \text{ form}]$$

$$= e^{\lim_{x\to 0}}\left[\frac{1+\tan x}{1-\tan x}-1\right]\frac{1}{x}$$

$$= e^{\lim_{x\to 0}\left(\frac{2\tan x}{1-\tan x}\right)\frac{1}{x}}$$

$$= e^{2\lim_{x\to 0}\frac{\tan x}{x}\cdot\lim_{x\to 0}\frac{1}{1-\tan x}} \left[\because \lim_{x\to 0}\frac{\tan x}{x}=1\right]$$

$$\therefore \qquad k = e^{2(1)\left(\frac{1}{1-0}\right)} = e^2$$

7.Sol: $\because \qquad f'(0^-) = \lim_{h\to 0} f(0-h)$

$$\Rightarrow \qquad f'(0^{-1}) = \lim_{h\to 0}\frac{e^{-1/h}-1}{e^{1/-h}+1}$$

$$\therefore \qquad f'(0^{-1}) = -1$$

and $\qquad f'(0^+) = \lim_{h\to 0}(0+h) = \lim_{h\to 0}\frac{e^{1/h}-1}{e^{1/h}-1}$

$$\Rightarrow \qquad f'(0^+) = \frac{1-0}{1+0} = 1$$

8.Sol: Given,

$$f(x) = \begin{cases} \log(\sec^2 x)^{\cot^2 x}, & \text{for } x \neq 0 \\ K, & \text{for } x = 0 \end{cases}$$

is continuous at $x=0$

$$f(0) = \lim_{x\to 0}\log(\sec^2 x)^{\cot^2 x}$$

$$= \lim_{x\to 0}\cot^2 x \log(\sec^2 x)$$

$$= \lim_{x\to 0}\frac{1}{\tan^2 x}\log(1+\tan^2 x)$$

$$= \lim_{x\to 0}\frac{\log(1+\tan^2 x)}{\tan^2 x}$$

$$= \lim_{\tan x\to 0}\frac{\log(1+\tan^2 x)}{\tan^2 x} = 1$$

9.Sol: Given, $f(x) = \begin{cases} ax^2+b, & b\neq 0, x\le 1 \\ bx^2+ax+c, & x>1 \end{cases}$

$$f(x) = \begin{cases} 2ax, & b\neq 0, x\le 1 \\ 2bx+a, & x>1 \end{cases}$$

Since, $f(x)$ is continuous at $x=1$

$$\therefore \qquad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$$

$$\Rightarrow \qquad a+b = b+a+c$$

$$\Rightarrow \qquad c = 0$$

Also, $f(x)$ is differentiable at $x=1$.

$\therefore$ (LHD at $x=1$) = (RHD at $x=1$)

$$\Rightarrow \qquad 2a = 2b(1)+a$$

$$\Rightarrow \qquad a = 2b$$

10.Sol: Given, $f(x) = \begin{cases} e^{-x}, & x\ge 0 \\ e^{x}, & x<0 \end{cases}$

$$\text{LHL} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^x = 1$$

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} e^{-x} = 1$$

Also, $\qquad f(0) = e^0 = 1$

$$\because \qquad LHL = RHL = f(0)$$

$\therefore$ It is continuous for every value of x.

Now, LHD at $x=0$

$$\left(\frac{d}{dx}e^x\right)_{x=0} = \left[e^x\right]_{x=0} = e^0 = 1$$

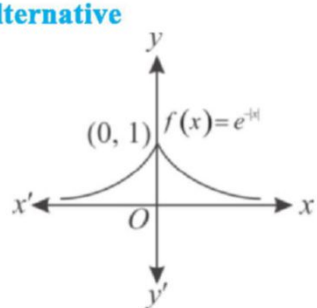
RHD at $x=0$

$$\left(\frac{d}{dx}e^{-x}\right)_{x=0} = \left[-e^{-x}\right]_{x=0} = -1$$

So, $f(x)$ is not differentiable at $x=0$

Hence, $f(x) = e^{-|x|}$ is continuous everywhere but not differentiable at $x=0$.

Alternative



Hence, it is clear from the figure that $f(x)$ is continuous everywhere and not differentiable at $x=0$.

11.Sol: We have,

$$f(x)=[x]+\left[x+\frac{1}{2}\right]=\begin{cases}0, & \text{if } 0<x<\frac{1}{2}\\ 1, & \text{if } x=\frac{1}{2}\\ 1, & \text{if } \frac{1}{2}<x<1\end{cases}$$

Clearly, $f(x)$ is discontinuous at $x=\frac{1}{2}$.

Also, $\lim\limits_{x\to 1/2^-} f(x)=0$

and $\lim\limits_{x\to \frac{1}{2}^+} f(x)=1$

12.Sol: Since, $|x-1|$ is not differentiable at $x=1$. So, $f(x)=|x-1|e^x$ is not differentiable at $x=1$.

Hence, the required set is $R-\{1\}$.

13.Sol: We have $f(x)=[x]\cos\left[\frac{2x-1}{2}\right]\pi$

Since, $g(x)=[x]$ is always discontinuous at all integral values of points. Here, $f(x)$ is discontinuous for all integral points.

14.Sol: $f(x)$ is discontinuous at points

where $\frac{\pi}{x+1}=(2m+1)f(x)\frac{\pi}{2}, m\in I$

$$\frac{x}{x+1}=\frac{2m+1}{2}$$

$$\Rightarrow \quad \{2-(2m+1)\}x=2m+1$$

$$(1-2m)x=(2m+1)$$

$$\therefore \quad x=\frac{2m+1}{1-2m}, m\in I$$

15.Sol: Given,

$$f(x)=\begin{cases}\dfrac{2\sin x-\sin 2x}{2x\cos x}, & \text{if } x\neq 0\\ a, & \text{if } x=0\end{cases}$$

Now $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\frac{2\sin x-\sin 2x}{2x\cos x}$

$$\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{2\cos x-2\cos 2x}{2(\cos x-x\sin x)}$$

$$=\frac{2-2}{2(1-0)}=0$$

Since, $f(x)$ is continuous at $x=0$

$$f(0)=\lim_{x\to 0} f(x)\Rightarrow a=0$$

16.Sol: $f(x)=\left(\frac{\log_e(1+ax)-\log_e(1-bx)}{x}\right)$

For $f(x)$ to be continuous at $x=0$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$$

$$\therefore \lim_{x\to 0^-}\left(\frac{\log_e(1+ax)-\log_e(1-bx)}{x}\right)$$

$$=\lim_{h\to 0}\frac{\log(1-ah)-\log(1+bh)}{-h}$$

Applying L' Hospital Rule, we get

$$=\lim_{h\to 0}\frac{\frac{-a}{1-ah}-\frac{b}{1+bh}}{-1}$$

$$=\lim_{h\to 0}\frac{a}{1-ah}+\frac{b}{1+bh}=a+b$$

$\therefore f(0)=a+b$ for continuous at $x=0$

17.Sol: We know that function $|x|$ is not differentiable at $x=0$

$\therefore \ |x^2-3x+2|=|(x-1)(x-2)|$

Hence, it is not differentiable at $x=1$ and 2

Now, $f(x)=(x^2-1)|x^2-3x+2|+\cos|x|$

differentiable at $x=2$.

For $1<x<2, f(x)=-(x^2-1)(x^2-3x+2)$
$+\cos x$

For $2<x<3, f(x)=(x^2-1)(x^2-3x+2)$
$+\cos x$

$$Lf'(x)=-(x^2-1)(2x-3)-2x(x^2-3x+2)-\sin x$$

$$Lf'(2)=-3-\sin 2$$

$$Rf'(x)=(x^2-1)(2x-3)+2x(x^2-3x+2)-\sin x$$

$$Rf'(2)=(4-1)(4-3)+0-\sin 2=3-\sin 2$$

Hence, $\quad Lf'(2)\neq Rf'(2)$.

So, $f(x)$ is not differentiable at $x=2$.

18.Sol: Given that $f''(x)-g''(x)=0$

Integrating, we get

$$f'(x)-g'(x)=c \qquad (1)$$

Put $x=1$

$$f'(1)-g'(1)=c$$

$\Rightarrow \qquad c=2$

Substituting the value of c in (1), we get

$$f'(x)-g'(x)=2 \qquad (2)$$

Integrating, (2), we get, $c=2$

$$f(x)-g(x)=2x+c' \qquad (3)$$

put $x=2$

$$f(2)-g(2)=4+c'$$

$\Rightarrow \qquad c'=2$

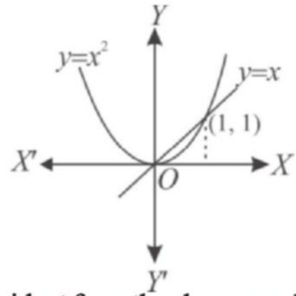
Substituting c' in (3)

$$f(x)-g(x)=2x+2$$

$$f\left(\frac{3}{2}\right)-g\left(\frac{3}{2}\right)=2\times\frac{3}{2}+2=5$$

19.Sol: Rewrite the given function as

$$h(x)=\min(x,x^2)=f(x)=\begin{cases} x, & x<0 \\ x^2, & 0\le x<1 \\ x, & x\ge 1 \end{cases}$$



It is evident from the above graph that the given function is continuous for all x.

Since, there are sharp edges at $x=0$ and $x=1$, the function is not differentiable at these points

Also, at $x\ge 1$, the function represents a straight line having slope 1, therefore $h'(x)=1, \forall x\ge 1$.

20.Sol: Given that $f(x)=\begin{cases} x^2 & \text{for } x<1 \\ 2-x & \text{for } x\ge 1 \end{cases}$

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=f(1)$$

$$\therefore \qquad \lim_{x\to 1^-} x^2=\lim_{x\to 1^+} 2-x=2-1$$

Hence, $f(x)$ is continuous at $x=1$

$$f'(x)=\begin{cases} 2x, & \text{for } x<1 \\ -1, & \text{for } x\ge 1 \end{cases}$$

$$\therefore \qquad \lim_{x\to 1^-} f'(x)=\lim_{x\to 1^-} 2x=2$$

$$\lim_{x\to 1^+} f'(x)=\lim_{x\to 1^+}(-x)=-1$$

$$\Rightarrow \qquad \lim_{x\to 1^-} f'(x)\neq\lim_{x\to 1^+} f'(x)$$

i.e., $\qquad$ L.H.D $\neq$ R.H.D

Hence, $f(x)$ is not differentiable at $x=1$.

21.Sol: Given, $\lim_{x\to 0^+} f(x)=\lim_{h\to 0}\cos(h)=1$

and $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} -[\cos(-h)]$

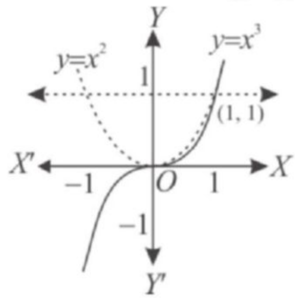
$= \lim_{h\to 0} -(\cosh) = -1$

So, $f(x)$ is continuous at $x = 0$.

22.Sol: It is evident from the graph of $f(x)$ that

$$f(x) = \begin{cases} 1, & x \ge 1 \\ x^3, & x < 1 \end{cases}$$

Clearly, $f(x)$ is everywhere continuous but it is not differentiable at $x = 1$.



23.Sol: Let $f(x) = \frac{\log_e(1 + x^2 \tan x)}{\sin x^3}$

This function is continuous at $x = 0$, then

$$\lim_{x\to 0} \frac{\log_e(1 + x^2 \tan x)}{\sin x^3} = f(0)$$

$$\Rightarrow \lim_{x\to 0} \frac{\log_e\left\{1 + x^2\left(x + \frac{x^3}{3} + \frac{2x^5}{15} + ...\right)\right\}}{x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - ...}$$

$$\Rightarrow \lim_{x\to 0} \frac{\log_e(1 + x^3)}{x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - ...}$$

[on neglecting higher power of x in $x^2 \tan x$]

$$\Rightarrow \lim_{x\to 0} \frac{x^3 - \frac{x^6}{2} + \frac{x^9}{3}}{x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - ...} = f(0)$$

$$\Rightarrow 1 = f(0)$$

24.Sol: At $x = 0, \pm 1$, the function is discontinuous.

At $x = 0, \log_e |x|$ is not defined

At $x = 1$ and $-1, \log_e |x|$ is 0, hence $f(x) = \infty$

So, $x = 0, \pm 1$ are the points of discontinuous.

25.Sol: Given that

$$\lim_{x\to 0} \frac{2^x - 2^{-x}}{x} = \lim_{x\to 0} 2^x \log 2 + 2^{-x} \log 2$$

$$= \log 2 + \log 2 = \log 4$$

Since, Function is continuous at $x = 0$.

$$\therefore \quad f(0) = \lim_{x\to 0} \frac{2^x - 2^{-x}}{x}$$

$$\Rightarrow \quad f(0) = \log 4$$

26.Sol: Given that $h(x) = f(g(x))$

upon differentiating, we get

$$h'(x) = f'(g(x) \cdot g'(x))$$

$$h'(3) = f'(g(3)) \cdot g'(3)$$

$$= f'(7) \cdot 6$$

$$= 2 \times 6 = 12$$

27.Sol: Given, $f : R \to R$ and

$$|f(x) - f(y)|^2 \le |x - y|^3$$

$$\Rightarrow \quad \frac{|f(x) - f(y)|^2}{|x - y|^2} \le |x - y|$$

Let, $\quad y = x + h, h \to 0$

$$\Rightarrow \lim_{h\to 0} \frac{|f(x) - f(x+h)|^2}{|x - (x+h)|^2} \le \lim_{h\to 0} |x - (x+h)|$$

$$\Rightarrow \lim_{h\to 0} \frac{|f(x+h) - f(x)|^2}{|h|^2} \le \lim_{h\to 0} |h|$$

$$\Rightarrow \quad |f'(x)|^2 \le 0$$

$$\Rightarrow \quad |f'(x)|^2 = 0$$

$$\therefore \quad f'(x) = 0$$

28.Sol: Given, $g(x)=f^{-1}(x)$

$$\Rightarrow \quad f\{g(x)\}=x$$

On differentiating, w.r.t. x, we get

$$f'\{g(x)\}\cdot g'(x)=1$$

$$\Rightarrow \quad g'(x)=\frac{1}{f'\{g(x)\}} \qquad (1)$$

$$\because \quad f'(x)=\frac{1}{1+x^3}$$

$$\because \quad f'\{g(x)\}=\frac{1}{1+\{g(x)\}^3}$$

Now, from Eq. (1), we get

$$g'(x)=1+\{g(x)\}^3$$

29.Sol: Given that $f(x^3)=4x^4 \forall x>0$

Let $x^3=t \Rightarrow x=t^{1/3}$

$$\therefore \quad f(t)=4t^{4/3}$$

On differentiating w.r.t. t, we get

$$f'(t)=4\cdot\frac{4}{3}(t)^{\frac{4}{3}-1}=4\cdot\frac{4}{3}(t)^{1/3}$$

$$\therefore \quad f'(x^3)=\frac{16}{3}(x^3)^{1/3}=\frac{16}{3}x$$

$$\therefore \quad f'(8)=f'(2^3)=\frac{16}{3}\times 2=\frac{32}{3}$$

30.Sol: Given that $\lim_{x\to 9}\frac{\sqrt{f(x)}-3}{\sqrt{x}-3}$

Since $\frac{0}{0}$ form, using l-Hospital rule, we get

$$=\lim_{x\to 9}\frac{\frac{f'(x)}{2\sqrt{f(x)}}}{\frac{1}{2\sqrt{x}}}$$

$$=\lim_{x\to 9}\frac{\sqrt{x}f'(x)}{\sqrt{f(x)}}$$

$$=\frac{\sqrt{9}f'(9)}{\sqrt{f(9)}}$$

$$=\frac{3\times 0}{3}=0$$

31.Sol: We have $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Since $f(x+y)=f(x)f(y)$

$$=\lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h}$$

$$=f(x)\lim_{h\to 0}\left(\frac{1+\sin(3h)g(h)-1}{h}\right)$$

$$=f(x)\lim_{h\to 0}\frac{\sin 3h}{3h}\lim_{h\to 0}g(h)$$

$$=f(x)\times 1\times g(0)=f(x)g(0)$$

32.Sol: We have, $2^x+2^y=2^{x+y}$

On differentiating w.r.t. x, we get

$$2^x\log 2+2^y\log 2\frac{dy}{dx}=2^{x+y}\log 2\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \quad 2^x+2^y\frac{dy}{dx}=2^{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{2^x-2^{x+y}}{2^{x+y}-2^y}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(1,1)}=\frac{2-4}{4-2}=-1$$

33.Sol: We have, $f(x)=x^2+e^x$

$$\Rightarrow \quad f_1(x)=2x+e^x$$

$$\Rightarrow \quad f_2(x)=2+e^x$$

$$\Rightarrow \quad f_3(x)=e^x \Rightarrow f_4(x)=e^x$$

Since, $f_3(x)=f_4(x)$

$\therefore$ Least value of n is 3.

34.Sol: We have,

$$f'(x)=g(x+1)$$

$$\Rightarrow \quad f''(x)=g'(x+1)$$

But $\quad g'(x)=h(x-1)$

$$\Rightarrow \quad g'(x+1)=h(x+1-1)$$
$$=h(x)$$
$$\therefore \quad f''(x)=h(x)$$
$$\Rightarrow \quad f''(2x)=h(2x)$$

35.Sol: Given that $y=\dfrac{x}{x+1}+\dfrac{x+1}{x}$

$$\Rightarrow \frac{dy}{dx}=\frac{(1)(x+1)-x(1)}{(x+1)^2}+\frac{(1)(x)-(x+1)\cdot 1}{x^2}$$
$$=\frac{1}{(x+1)^2}-\frac{1}{x^2}$$
$$\Rightarrow \quad \frac{d^2y}{dx^2}=\frac{-2}{(1+x)^3}+\frac{2}{x^3}$$

On putting $x=1$, we get

$$\left(\frac{d^2y}{dx^2}\right)_{x=1}=\frac{-2}{(1+1)^3}+\frac{2}{(1)^3}$$
$$=\frac{-2}{8}+2$$
$$=-\frac{1}{4}+2=\frac{7}{4}$$

36.Sol: Given, $x=a\cos^3\theta$ and $y=a\sin^3\theta$

On differentiating both sides w.r.t. θ, we get

$$\frac{dx}{d\theta}=3a\cos^2\theta(-\sin\theta)$$
and
$$\frac{dy}{d\theta}=3a\sin^2\theta(\cos\theta)$$
Now,
$$\frac{dy/d\theta}{dx/d\theta}=\frac{3a\sin^2\theta(\cos\theta)}{3a\cos^2\theta(-\sin\theta)}$$
$$\Rightarrow \quad \frac{dy}{dx}=-\tan\theta$$
$$\therefore \quad 1+\left(\frac{dy}{dx}\right)^2=1+(-\tan\theta)^2$$
$$=1+\tan^2\theta$$
$$=\sec^2\theta$$

37.Sol: Given that $y=(\sin^{-1}x)^2$

Differentiating on both sides, we get

$$\frac{dy}{dx}=2\sin^{-1}x\cdot\frac{1}{\sqrt{1-x^2}}$$
$$\Rightarrow \quad \sqrt{1-x^2}\frac{dy}{dx}=2\sin^{-1}x$$

Differentiating on both sides,

$$\sqrt{(1-x^2)}\frac{d^2y}{dx^2}-\frac{2x}{2\sqrt{1-x^2}}\frac{dy}{dx}=\frac{2}{\sqrt{1-x^2}}$$
$$2(1-x^2)\frac{d^2y}{dx^2}-2\frac{dy}{dx}\cdot x=4$$
$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=2$$

38.Sol: Given that,

$$x=a(1+\cos\theta) \qquad (1)$$
and
$$y=a(\theta+\sin\theta) \qquad (2)$$

Differentiating eq. (1) and (2) w.r.t. θ, we get

$$\frac{dx}{d\theta}=-a\sin\theta$$
and
$$\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{a(1+\cos\theta)}{-a\sin\theta}$$
$$\Rightarrow \quad \frac{dy}{dx}=\frac{1+\cos\theta}{-\sin\theta}$$
$$=-\frac{1+2\cos^2\frac{\theta}{2}-1}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}=-\frac{cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$
$$=-\cot\frac{\theta}{2}$$
$$\therefore \quad \frac{d^2y}{dx^2}=\operatorname{cosec}^2\frac{\theta}{2}\cdot\frac{1}{2}\frac{d\theta}{dx}$$
$$=-\frac{1}{2}\frac{1}{a}\frac{\operatorname{cosec}^2\frac{\theta}{2}}{\sin\theta}$$

$$\therefore \quad \left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{2}} = -\frac{1}{a}$$

39.Sol: $f'(x) = \dfrac{f(b)-f(a)}{b-a} = \dfrac{16-(-2)}{6-0}$

$$= \frac{18}{6} = 3$$

40.Sol: Given, $f(x) = \begin{cases} 2x^2-1, & \text{if } 1 \le x \le 4 \\ 151-30x, & \text{if } 4 < x \le 5 \end{cases}$

Differentiability at $x = 4$,

$$LHD = \lim_{h\to 0} \frac{f(4-h)-f(4)}{-h}$$

$$= \lim_{h\to 0} \frac{2(4-h)^2 - 1 - (2\times 16 - 1)}{-h}$$

$$= \lim_{h\to 0} \frac{2(16+h^2-8h)-32}{-h}$$

$$= \lim_{h\to 0} \frac{2h(h-8)}{-h}$$

$$= \lim_{h\to 0} \left[-2(h-8)\right] = 16$$

$$RHD = \lim_{h\to 0} \frac{f(4+h)-f(4)}{h}$$

$$= \lim_{h\to 0} \frac{151-30(4+h)-(2\times 16-1)}{h}$$

$$= \lim_{h\to 0} -\left(\frac{30h}{h}\right) = -30$$

$$\therefore \quad LHD \ne RHD$$

Hence, $f(x)$ is not differentiable at $x = 4$.

41.Sol: Given, $f(x) = x^3 - x + p, \quad (0 \le x \le 2)$

Here, $\quad a = 0, b = 2$

$$f'(c) = \frac{f(b)-f(a)}{b-a} \qquad (1)$$

$$\Rightarrow \quad f'(c) = \frac{f(2)-f(0)}{2-0}$$

$$\therefore \quad 3c^2 - 1 = \frac{(2)^3 - 2 + p - (0^2 - 0 + p)}{2-0}$$

$$\Rightarrow \quad 3c^2 - 1 = \frac{6}{2} = 3$$

$$\Rightarrow \quad 3c^2 = 4$$

$$\Rightarrow \quad c = \pm\frac{2}{\sqrt{3}} = \frac{\pm 2\sqrt{3}}{3}$$

$$\therefore \quad c = \frac{2\sqrt{3}}{3}$$

42.Sol: By Lagrange mean value theorem

$$f'(x) = \frac{f(5)-f(1)}{5-1} \ge 9$$

$$\Rightarrow \quad \frac{f(5)+3}{4} \ge 9$$

$$\Rightarrow \quad f(5) \ge 36 - 3$$

$$\Rightarrow \quad f(5) \ge 33$$

43.Sol: Given, $f(x) = [x], x \in (-3.5, 100)$

As, we know greatest integer is discontinuous on integer values.

In the given interval, the integer values are (-3,-2,-1,0,...,99)

Hence, total number of integers is 103.

44.Sol: $f'(1+0) = \lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$

$$= \lim_{h\to 0} \frac{\frac{[1-h]}{|1-h|} - \frac{[1]}{1}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{1+h} - 1}{h} = \lim_{h\to 0} \frac{-h}{h(1+h)} = -1$$

$$f'(1-0) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0} \frac{\frac{[1+h]}{|1+h|} - \frac{[1]}{1}}{h}$$

$$= \lim_{h\to 0} \frac{0-1}{-h} = \infty$$

45.Sol: Given that $f(x) = e^x.g(x)$

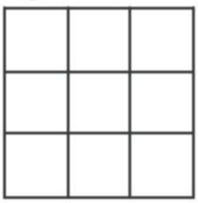
$$\Rightarrow f'(x) = e^x.g'(x) + e^x.g(x)$$

$$\Rightarrow f'(0) = e^0.g'(0) + e^0.g(0) = 2 + 4 = 6$$

NSEJS

[2014-2015]

1. Figure shows a square gird of order 3, which of the following is correct formula for the total number of squares in a similar grid of order n.



(a) $\dfrac{n(n+1)}{2}$ (b) $\dfrac{n^2(n+1)^2}{4}$

(c) $\dfrac{n(n+1)(2n+1)}{6}$ (d) $\dfrac{n(n+1)(n+2)}{6}$

2. A water filter advertisement claims to provide 8 liters of water per hour. How much time does it take to fill four bottles of 1.5 liters each?

(a) 2 hr (b) 1 hr
(c) 30 min (d) 45 min

3. If set of marbles, of radius 5 cm, is poured into a cube of side 1m. The maximum number of marbles that can be filled into the box are

(a) 2000 (b) 1000
(c) 1500 (d) 3000

4. A round table cover has six equal designs as shown in the adjacent figure. If the radius of the cover is 4cm, then cost of making the designs at the rate of Rs 10.00 per cm^2 (round off your answer to a nearest rupee) is

(a) Rs. 85 (b) Rs. 86
(c) Rs. 90 (d) Rs. 87

5. The houses of a row are numbered consecutively from 1 to 49. Find the value of x such that the sum of the numbers of houses preceding the house numbered x is equal to the sum of the numbers of the houses following it.

(a) 25 (b) 35
(c) 37 (d) No such value exists

6. A piece of wire 60 cm long is cut into parts, one of them being 24 cm long. Each part is then bent to form a square. The ratio of the area of the larger to the smaller square is :

(a) 9/4 (b) 7/4 (c) 3/2 (d) 11/2

7. Scientist in an R & D company made three design improvements on a car : the first saves 50% of fuel, the second saves 30% of fuel and the third saves 20%. If the company implements all three design changes at once, the new car will consume fuel that is - % of the fuel consumption of normal car

(a) 50% (b) 100% (c) 28% (d) 20%

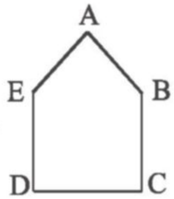
8. The adjacent figure is a modification of the Switzerland flag to suit the problem! five identical small squares from the central cross. the length of each side of the big square is 10 m. If the area of the white cross is 20% of the area of the flag. then the length of the side of the small square is :

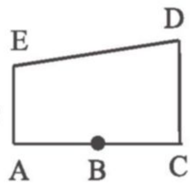
(a) 2m (b) 2.25 (c) 1.6 m (d) 1.75 m

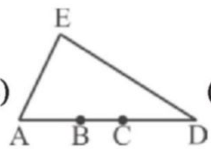
9. A number is said to be triangular number if it is the sum of consecutive numbers beginning with I. Which one of the following is not a triangular number :

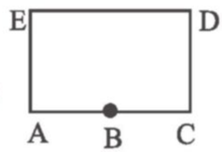
(a) 1431 (b) 190 (c)28 (d) 506

10. If the distance between A and B is 230 km, B and C is 120 km, C and A is 350 km. Also, if the distance between C and D is 200 km, distance between D and B is 330 km and distance from A to E is 100 km and distance between D and E is 570 km. The diagram (not drawn to sclae) that represents this graphically is

(a) 

(b) 

(c) 

(d) 

11. The sum of 2 digits x and y is divisible by 7. What can one say about a 3 digit number located by these two digits
(a) xxy is divisible by 7
(b) xyx is divisible by 7
(c) xyx is divisible by 7^2
(d) yyx is divisible by 7

12. A number of bacteria are placed in a glass. 1 second later each bacterium divides in three, the second each of the resulting bacteria divides in three again, and so on. After one minute the glass full. When was 1/9th of the glass full?
(a) 15 sec (b) 45 sec (c) 58 sec (d) 38 sec

13. A number x is a rational number if there exists integers p and q such that $x = p/q$. This is definition of rational numbers in which,
(a) Both p & q can be zero
(b) Both p & q should not be zero
(c) q can be zero but not p
(d) p can be zero but not q

14. The least positive integer, n, such that 2 divides n, 3 divides n + 1, 4 divides n + 2, 5 divides n + 3 and 6 divides n + 4 is
(a) 62 (b) 120 (c) 720 (d) 52

15. The solution set of the inequality

$0 < \dfrac{x}{x+1} < 1, x \in \mathrm{R}$ is

(a) Set of all positive real numbers
(b) Set of all real numbers except –1
(c) Set of all non-negative real numbers
(d) Set of all numbers satisfying $0 \le x \le 1, x \in \mathrm{R}$

16. Number plate of a vehicle consists of 4 digits. The first digit is the square of second. The third digit is thrice the second and the fourth digit is twice the second. The sum of all 4 digits is thrice the first. The number is
(a) 1132 (b) 4264 (c) 1642 (d) 9396

17. If the highest common factor of a, b and c is 1, where a, b and c belong to the set of natural numbers, then the highest common factor of $(a \times b)$ and c is:
(a) c (b) $a \times b$
(c) 1 (d) Insufficient data

ANSWER KEY

1. c	2. d	3. b	4. d	5. b
6. a	7. c	8. a	9. d	10.
11. b	12. c	13. d	14. a	15. a
16. d	17. d			

HINTS & SOLUTIONS

1.Sol: There are $n \times n = n^2$ squares of size 1×1.

Like wise $(n-1) \times (n-1) = (n-1)^2$ squares of size 2×2. .... and so on. That is, total number of squares in $n \times n$ grid is

$$1^2 + 2^2 + 3^2 + \ldots\ldots + (n-1)^2 + n^2$$

$$= \frac{n(n+1)(2n+1)}{6}$$

2.Sol: Given that, A water filter provides 8 liters of water per hour. To fill 4 bottles of each 1.5

litres of water we need total $6\ lts$. That is

$6\ lts$ can be filled in $\frac{6}{8}=\frac{3}{4}$ hours.

3.Sol: Since diameter of each marble is 10cm. So the cube can be filled by marbles in 10 layers as length of cube is 100cm. Therefore total number of marbles can be filled in the cube is $10\times 100=1000$.

4.Sol: Let us look at the design in the given figure. We see that,

Total area of design = Area of a circle - Area of hexagen

$$=\pi(4)^2-6\times\frac{\sqrt{3}}{4}(4^2)$$

$$=8.71cm^2$$

Cost of making $1cm^2$ is $Rs10/-$

Cost of making $8.71cm^2$ design is

$$8.71\times 10=87.1$$

5.Sol: Sum of numbers of houses preceding x is equal to sum of the number houses following

$$S_{x-1}=S_{49}-S_x$$

$$\frac{(x-1)(x-1+1)}{2}=\frac{49(49+1)}{2}-\frac{x(x+1)}{2}$$

i.e., $$\frac{x(x-1)}{2}=\frac{49\times 50-x(x+1)}{2}$$

$$\Rightarrow x(x-1)=2450-x(x+1)$$

$$\Rightarrow x^2-x+x^2+x=2450$$

i.e., $$2x^2=2450$$

$$\Rightarrow x^2=1225$$

$$x=\sqrt{1225}$$

$$=35$$

$$\therefore x=35$$

6.Sol: Since the piece of wire 60cm long is cut into two parts, one of them being 24cm long and the other is 36 cm. That is perimeter of larger square and smaller square are 36 cm and 24cm respectively. That is

$P_\ell=4S_1=36cm$ and $P_S=4S_2=24cm$

$\therefore$ Area of larger square is S_1^2 and like wise smaller square is S_2^2.

Therefore its ratio is $S_1^2:S_2^2$

i.e., $9^2:6^2=9:4$.

7.Sol: Let the initial fuel be 100 *lts*, after first design used. It reduces to 50 *lts* of fuel consumption, likewise based on using other two designs, the fuel consumption reduces to 28 lts. that is 28 %.

8.Sol: Let the side of a small square be '*a*'. and length of each side of the big square is *10M*. Since the area of cross in 20% of the area of big square. that is

$$A_C=20\%A_B$$

i.e., $$A_C=20\%\text{ of }100$$

i.e., $$A_C=20.$$

$$5a^2=20$$

$$\Rightarrow a=\pm 2$$

9.Sol: $1+2+3+.....+n=\frac{n(n+1)}{2}$

(a) $\frac{n(n+1)}{2}=1431$

$n(n+1)=2862$

$n=53$

(b) $\frac{n(n+1)}{2}=190$

$n(n+1)=380$

$n=19,\ n+1=20$

(c) $\frac{n(n+1)}{2}=28$

$n(n+1)=56$

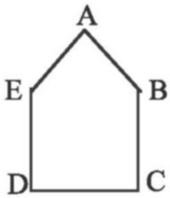
$n=7$

(d) $\frac{n(n+1)}{2}=506$

$n(n+1)=1012$

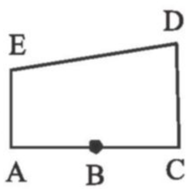
no natural value of n possible

10.Sol: (a)



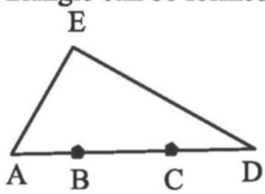
$AB + BC = AC$
$230 + 120 = 350$
A, B, C are collinear

(b), (d)



$BD > BC + CD$
$330 > 120 + 200$
No triangle can be formed

(c)



$BD \neq BC + CD$
$330 \neq 120 + 200$
B, C, D are not collinear

11.Sol: Given that $x + y = 7k, k \in z$

$\Rightarrow \qquad y = 7k - x$

Now, we take each option, and verify option (a) is rewritten as

$$x(100) + x(10) + y = x(110) + 7k - x$$
$$= 109x + 7k$$

which is not divisible by 7.
Now, option (b), is rewritten as

$$x(100) + y(10) + x = (101)x + 10(7k - x)$$
$$= 91x + 70k$$

Which is divisble by 7.

12.Sol: Bacteria becomes thrice of its population every second, and it took them 60 sec to fill glass.

Time taken by bacteria to fill $\frac{1}{3}$ rd of glass is time taken by bacteria to fill full glass -1.

Like wise time taken to fill $\frac{1}{9}$ th of a glass is $60 - 2 = 58$ sec.

13.Sol: Conceptual

14.Sol: Clearly $n-2$ is divisble by 3, 4, 5, and 6. That is, $(n-2)$ is the LCM of (3, 4, 5, 6). So $n - 2 = 60$. Therefore $n = 60 + 2 = 62$.

15.Sol: Rewriting the given inequality as

$\frac{x}{x+1} > 0$ and $\frac{x}{x+1} < 1$

for $\qquad \frac{x}{x+1} > 0$

i.e., either $x > 0$ and $x + 1 > 0$

$\Rightarrow \qquad x > 0 \qquad (1)$

either $x < 0$ and $x + 1 < 0$

i.e., $\qquad x < -1 \qquad (2)$

now $\qquad \frac{x}{x+1} < 1$

$\Rightarrow \qquad 1 - \frac{x}{x+1} > 0$

i.e., $\qquad \frac{1}{x+1} > 0$

i.e., $\qquad x + 1 > 0$

$\therefore \qquad x > -1 \qquad (3)$

from (1), (2) and (3), we get $x > 0$.

16.Sol: Let the 4-digit number be *abcd*. Such that $a = b^2, c = 3b, d = 2b$

and $a + b + c + d = 3a$.

i.e., $\qquad b + c + d = 2a$

$$b + 3b + 2b = 2(b^2)$$

$$6b = 2b^2$$

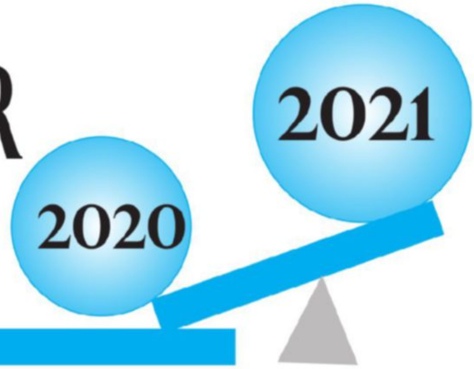
$\Rightarrow b = 0$ or $b = 3$

Now $b = 3$ gives us $a = 9, c = 9,$ and $d = 6$

$\therefore$ The required number is 9396.

17.Sol: In sufficient data

MOCK TEST PAPER



JEE MAIN - 1

Section - I (Multiple Choice Questions)

1. Let $2A+B=\begin{bmatrix}1 & 0 & 3\\ -1 & 4 & 6\\ 2 & 5 & 2\end{bmatrix}$, $A\text{-}2B=\begin{bmatrix}2 & -1 & 5\\ 0 & 3 & 6\\ 1 & 2 & 1\end{bmatrix}$.

Then Tr (A)-Tr (B) has the value equal to (where Tr (A) denotes the trace of matrix A)

(1) 3 (2) 5 (3) 6 (4) 7

2. If $f(x)$ is differentiable & satisfying

$f(x+y)=f(x)+f(y)+(x^2+y^2)xy$, $\forall\, x,y \in R$ and

$\lim\limits_{x\to 0}\frac{f(x)}{x}=2$ then $f(x)$ will be

(1) $2+x^3$ (2) $2x+\frac{x^4}{4}$

(3) $2+\frac{x^4}{4}$ (4) $2x+x^4+2$

3. $\int\frac{(\cot^2 x-n+1)}{\cos^n x}du=-f(x)\left[[f(x)]\right]^2+c$ where

$f\left(\frac{\pi}{2}\right)=1$ then minimum value of

$\left[[f(x)]^2+[g(x)]^2\right]^n$

(1) 1 (2) 2^n (3) 2^{2n} (4) 2^{4n}

4. $\int\limits_{-\infty}^{0} e^{122x}(1-e^x)^{19}\,dx$ is equal to

(1) $\frac{19!121!}{71!}$ (2) $\frac{19!121!}{70!}$

(3) $\frac{19!121!}{141!}$ (4) $\frac{19!121!}{140!}$

5. Perimeter of the locus represented by $\arg\left(\frac{z+i}{z-i}\right)=\frac{\pi}{4}$ (where $i=\sqrt{-1}$) is equal to

(1) $\frac{3\pi}{2}$ (2) $\frac{3\pi}{\sqrt{2}}$

(3) $\frac{\pi}{\sqrt{2}}$ (4) None of these

6. The value of p for which the sum of the squares of the roots of equation $x^2-(p+3)x+(5p-2)=0$ assume its least value is

(1) 1 (2) 2 (3) 3 (4) 4

7. How many 6 letter words can be formed using the letter from the word 'BORING ROAD' if each word has 3 vowels and 3 consonants

(1) 936 (2) 5616 (3) 17280 (4) 33676

8. Sum of the series

$S=1^2-2^2+3^2-4^2+...-2002^2+2003^2$ is

(1) 2007006 (2) 1005004

(3) 2000506 (4) None of these

9. If 7^{th} term from beginning in the binomial expansion $\left(\frac{3}{(84)^{1/3}}+\sqrt{3}\ln x\right)^9$, $x>0$ is equal to 729, then possible value of x is

(1) e^2 (2) e (3) $\frac{e}{2}$ (4) 2e

10. The complete set of real values of λ such that point $P(\lambda,\sin\lambda)$ lies inside the triangle formed by lines $x-2y+2=0$, $x+y=0$ and $x-y-\pi=0$ is

(1) $\left(\frac{\pi}{3},\frac{\pi}{2}\right)$ (2) $\left(0,\frac{\pi}{6}\right)\cup\left(\frac{\pi}{3},\frac{\pi}{2}\right)$

(3) $\left(0,\frac{\pi}{2}\right)\cup\left(\frac{2\pi}{3},\pi\right)$ (4) $(0,\pi)$

11. The image of the parabola $x^2=4y$ in the line $x+y=2$ is

(1) $(x-2)^2=4(2-y)$ (2) $(y-2)^2=4(2-x)$

(3) $(x-2)^2=4(2+y)$ (4) $(y-2)^2=4(2+x)$

12. $\frac{\sin A}{\sin C}=\frac{\sin(C+2B)}{\sin(A+2C)}$ then $a\sin A$, $b\sin B$, $c\sin C$ are in,

(1) AP (2) GP

(3) HP (4) None of these

13. Area of the ellipse $3x^2+4xy+3y^2+2x+8y+6=0$ is

(1) $\frac{\pi}{\sqrt{5}}$ (2) $\sqrt{5}\pi$ (3) $\frac{4\pi}{\sqrt{5}}$ (4) $4\sqrt{5}\,\pi$

14. A hyperbola having the transverse axis of length $2\sin\theta$, is confocal with the ellipse $3x^2+4y^2=12$. Then its equation is:

(1) $x^2\operatorname{cosec}^2\theta-y^2\sec^2\theta=1$

(2) $x^2\sec^2\theta-y^2\operatorname{cosec}^2\theta=1$

(3) $x^2\sin^2\theta-y^2\cos^2\theta=1$

(4) $x^2\cos^2\theta-y^2\sin^2\theta=1$

15. The sum of the solutions in $x\in(0,4\pi)$ of the equation $7\,\mathrm{Sin}\frac{x}{3}\left(\mathrm{Sin}\left(\frac{\pi+x}{3}\right)\right)\mathrm{Sin}\left(\frac{2\pi+x}{3}\right)=1$ is

(1) 6π (2) 4π

(3) 2π (4) None of these

16. This equation $\sum_{r=-1}^{m}\left(x-\frac{1}{2^r}\right)^{2n+1}=0$, has

(1) Only one real root (2) Has all real root

(3) Has 2n real root (4) has (2n+1) real root

17. Let *f(x)* be a continuous function such that $\int_{-3}^{3}f(x)dx=0$ and $\int_{0}^{3}f(x)dx=3$, then the area bounded by *y=f(x)*, x-axis, *x=-3* and *x=3* is equal to

(1) 1 (2) 3

(3) 6 (4) cannot be evaluated

18. If the line $x+2by+7=0$ is a diameter of the circle $x^2+y^2-6x+2y=0$, then *b*=

(1) 3 (2) -5 (3) 5 (4) -1

19. The contrapositive of 'If Kapil is rich then he is honest' is

(1) If Kapil is not rich then he is dishonest

(2) If Kapil is dishonest then he is not rich

(3) Kapil is not rich or he is dishonest

(4) Kapil is dishonest and not rich

20. If $f(-x)=f(x)$. Then $f'(x)$ must be

(1) an even function (2) an odd function

(3) a periodic function

(4) neither even nor odd

Section - II (Numerical Based Questions)

1. Two numbers *x* and *y* are randomly selected on the real number line, such that $0\le x,y\le 60$. Find the probability that the difference between selected number is greater than 10.

2. If line $\vec{r}=(i-2j-k)+\lambda(2i+j+2k)$ is parallel to the plane $\vec{r}.(3i-2j-mk)=14$, then the value of *m* is

3. Consider at three dimensional figure represented by $xyz^2=2$, then its minimum distance from origin is

4. Number of real value of *x* satisfy this equation, $2^{2x}+2^x(3-5\cos x)+1=0$

5. Let $\frac{df(x)}{dx}=\frac{e^{\sin x}}{x}, x>0$. If $\int_1^4\frac{3e^{\sin x^3}}{x}dx=f(k)-f(1)$, then $k=$ ____

ANSWER KEY

Section - I

1. 2	2. 2	3. 3	4. 3	5. 2
6. 2	7. 3	8. 1	9. 2	10. 4
11. 2	12. 1	13. 1	14. 1	15. 1
16. 1	17. 4	18. 3	19. 2	20. 2

Section - II

1. 0.30	2. 2	3. 2	4. 0	5. 64

HINTS & SOLUTIONS

Section - I

1.Sol: $2Tr(A)+Tr(B)=7$ and $Tr(A)-2Tr(B)=6$

$\Rightarrow Tr(A)=4$ and $Tr(B)=-1$

2.Sol: $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{f(h)+f\left(x^2+h^2\right)+xh}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{f(h)}{h}+x\left(x^2+h^2\right)=2+x^3$$

$$f(x)=2x+\frac{x^4}{4}+k$$

$$\lim_{x\to 0}\frac{f(x)}{x}=2\Rightarrow\frac{2x+\frac{x^4}{4}+K}{x}=2\Rightarrow k=0$$

$$so, f(x)=2x+\frac{x^4}{4}$$

3.Sol: $\int\frac{cosec^2x-n}{cos^nx}dx$

$$\int\left(cosec^2x-n\right)\sec^n xdx$$

$$=\int\left(cosec^2x\right)\sec^n xdx-\int nsec^n xdx$$

$$=-\sec^{(n-1)}xcosecx$$

so, $f(x)=cosecx$ and $g(x)=\sec x$

$\therefore$ Minimum value of $\sqrt{\sec^2x+\operatorname{cosec}^2x}=4$

So $(4)^n=(2)^{2n}$.

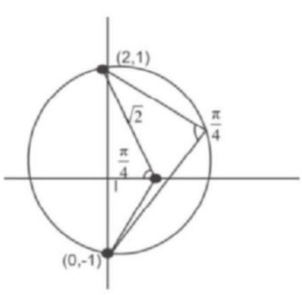
4.Sol: Put $e^x=t$

$$\Rightarrow\int_0^1 t^{121}(1-t)^{19}\,dt$$

$$l_{121,18}=\frac{1}{(121+19+1)\left({}^{121+19}C_{19}\right)}$$

$$=\frac{1}{141\times\left({}^{140}C_{19}\right)}=\frac{19!121!}{141!}$$

5.Sol:



Plotting the locus on argand plane, we get

$$\therefore\text{Perimeter}=\frac{3}{4}\times 2\pi.\sqrt{2}=\frac{3\pi}{\sqrt{2}}$$

6.Sol: Let roots are

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=p^2-4p+13$$

$$=(p-2)^2+9$$

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=p^2-4p+13$$

$$=(p-2)^2+9$$

$\alpha^2+\beta^2$ is minimum when $p=2$

7.Sol: 3 vowels can be selected by = 12 ways

Consonant can be selected by 72 ways

Total no of ways is $={}^6C_3\times 12\times 72=17280$

8.Sol: We can write S as

$$S=(1-2)(1+2)+(3-4)(3+4)+...$$

$$+(2001-2002)(2001+2002)+2003^2$$

$$=-[1+2+3+4+...+2002]+2003^2$$

$$=-\frac{1}{2}(2002)(2003)+2003^2$$

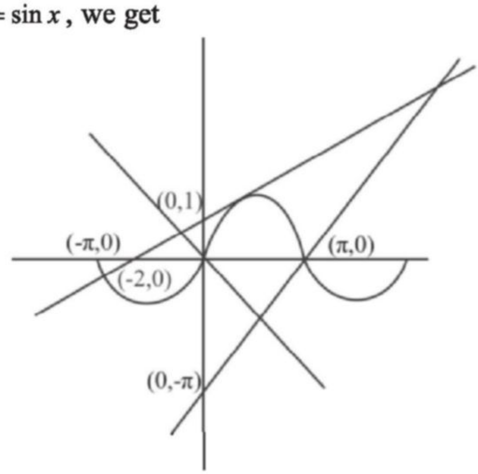
$$=-\frac{1}{2}(2002)(2003)+2003^2=2007006$$

9.Sol: $T_7={}^9C_6\left(\frac{3}{(84)^{1/3}}\right)^3\left(\sqrt{3}\ln x\right)^6=729$

$$\Rightarrow(\ln x)^6=1$$

$$\Rightarrow\ln x=\pm 1\Rightarrow x=e,\frac{1}{e}$$

10.Sol: Plotting the given lines and the curve $y=\sin x$, we get



Clearly, $\lambda\in(0,\pi)$

11.Sol: Let any point on $x^2 = 4y$ as $(2t^2, t^2)$ mirror image of $(2t^2, t^2)$ along line $x+y=2$ is (h,k) is

$$\frac{h-2t}{1} = \frac{k-t^2}{1} = -2\left(\frac{2t+t^2-2}{2}\right)$$

$$h = 2t - 2t - t^2 + 2 = 2 - t^2$$

$$k = t^2 - 2t - t^2 + 2 = 2 - 2t$$

$$\frac{(2-k)}{2} = t \Rightarrow t^2 = 2 - h$$

$$(2-k)^2 = 4(2-h) \Rightarrow (y-2)^2 = 4(2-x)$$

12.Sol: $\dfrac{\sin A}{\sin C} = \dfrac{\sin(A-B)}{\sin(B-C)}$

$$\sin(B+C)\sin(B-C) = \sin(A+B)\sin(A-B)$$

$$\sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$$

$\Rightarrow \sin^2 A, \sin^2 B, \sin^2 C$ are in AP

$\therefore \dfrac{a}{2R}\sin A, \dfrac{b}{2R}\sin B, \dfrac{c}{2R}\sin C$ also in AP

i,e., $a\sin A, b\sin B, c\sin C$ are in AP.

13.Sol: This is shifted ellipse

$$3(x-1)^2 + 4(x-1)(y+2) + 3(y+2)^2 = 0$$

$$3X^2 + 4XY + 3Y^2 = 1$$

so rotate 45^0 we get

$$\frac{x^2}{5} - \frac{y^2}{1} = 1 \Rightarrow A = \frac{\pi}{\sqrt{5}}$$

14.Sol: The given ellipse is

$$\frac{x^2}{4} + \frac{y^2}{3} = 1 \Rightarrow a = 2, b = \sqrt{3}$$

$$\therefore 3 = 4(1-e^2) \Rightarrow e = \frac{1}{2}$$

$$\therefore ae = 2 \times \frac{1}{2} = 1$$

Hence, the equation eccentricity e_1, of the hyperbola is given by

$$1 = e_1 \sin\theta \Rightarrow e_1 = \operatorname{cosec}\theta \Rightarrow b^2 = \sin^2\theta(\operatorname{cosec}^2\theta - 1)$$
$$= \cos^2\theta$$

Hence, equation of hyperbola is

$$\frac{x^2}{\sin^2\theta} - \frac{y^2}{\cos^2\theta} = 1 \text{ or } x^2\operatorname{cosec}^2\theta - y^2\sec^2\theta = 1$$

15.Sol: $7\left(\dfrac{\sin x}{4}\right) = 1 \Rightarrow \sin x = \dfrac{4}{7}$

$\therefore$ sum of solutions is 6π

16.Sol: $f(x) = \displaystyle\sum_{r=-1}^{m}\left(x - \frac{1}{2^r}\right)^{2n-1}$

After differentiation $f'(x)$ is positive so *f(x)* is increasing only one real root and other are imaginary.

17.Sol: Conceptual

18.Sol: Given $x + 2by + 7 = 0$ is a diameter of the given circle i.e, the centre of the circle passes through the diameter.

$$\Rightarrow 3 - 2b(1) + 7 = 0$$

i.e., $b=5$

19.Sol: If Kapil is dishonest then he is not rich.

20.Sol: $f(-x) = f(x) \quad \Rightarrow -f'(-x) = f'(x)$

$\therefore f'(x)$ is an odd function.

Section - II

1.Sol: $S = \{x, y : 0 \le x, y \le 60\}$

A $\rightarrow$ Event that $x - y \le 10$

B $\rightarrow$ Event $y - x \le 10$

$$\text{n(E)} = 60^2 - 2 \times \frac{1}{2} \times 50 \times 50$$

$$\text{P(E)} = \frac{1100}{3600} = 0.30$$

2.Sol: $\left(2\hat{i} + \hat{j} + 2\hat{k}\right).\left(3\hat{i} - 2\hat{j} - m\hat{k}\right) = 0$

$\Rightarrow 6 - 2 - 2m = 0$ or $m = 2$

3.Sol: Distance $= \sqrt{x^2 + y^2 + z^2}$

$$\therefore x^2 + y^2 + \frac{2}{xy} = x^2 + y^2 + \frac{1}{xy} + \frac{1}{xy} \ge 4$$

$\therefore$ Minimum distance = 2

4.Sol: Divide by 2^x,

$$2^x + 2^{-x} = 5\cos x - 3$$
$$2^x + 2^{-x} - 2 = 5(\cos x - 1)$$

We know L.H.S ≥ 0 and R.H.S ≤ 0

$$\cos x - 1 = 0 \text{ and } 2^x + 2^{-x} = 2$$

$\Rightarrow x = 0, 2\pi, 4\pi$ and $x = 0$

$$x = 0$$

5.Sol: $\dfrac{df(x)}{dx} = \dfrac{e^{\sin x}}{x} \Rightarrow \dfrac{df(x^3)}{d(x^3)} = \dfrac{e^{\sin x^3}}{x^3}$

$$\Rightarrow \frac{df(x^3)}{d(x^3)} \cdot \frac{d(x^3)}{d(x^3)} = \frac{e^{\sin x^3}}{x^3} \cdot 3x^2$$

$$\Rightarrow \frac{df(x^3)}{dx} = \frac{3e^{\sin x^3}}{x} \Rightarrow \int_1^4 \frac{3e^{\sin x^3}}{x}dx$$

$$= \left[f(x^3)\right]_{x=1}^4 = f(64) - f(1)$$

$\therefore \quad k = 64$

Previous years JEE MAIN Questions

TRIGONOMETRY

[OFFLINE QUESTIONS]

1. If the sum of all the solutions of the equation $8\cos(x).\left(\cos\left(\frac{\pi}{6}+x\right).\cos\left(\frac{\pi}{6}-x\right)-\frac{1}{2}\right)=1$ in $[0,\pi]$ is $k\pi$, then k is equal to : **[2018]**

(a) $\frac{20}{9}$ (b) $\frac{2}{3}$ (c) $\frac{13}{9}$ (d) $\frac{8}{9}$

2. If $5\left(\tan^2 x-\cos^2 x\right)=2\cos 2x+9$, then the value of $\cos 4x$ is : **[2017]**

(a) $-\frac{7}{9}$ (b) $-\frac{3}{5}$ (c) $\frac{1}{3}$ (d) $\frac{2}{9}$

3. If $0 \le x < 2\pi$, then the number of real values of which satisfy the equation $\cos x+\cos 2x+\cos 3x+\cos 4x=0$ **[2016]**

(a) 7 (b) 9 (c) 3 (d) 5

4. Let $f_k(x)=\frac{1}{k}\left(\sin^k x+\cos^k x\right)$ where $x \in R$ and $k \ge 1$. Then $f_4(x)-f_6(x)$ equals **[2014]**

(a) $\frac{1}{4}$ (b) $\frac{1}{12}$ (c) $\frac{1}{6}$ (d) $\frac{1}{3}$

5. The expression $\frac{\tan A}{1-\cot A}+\frac{\cot A}{1-\tan A}$ can be written as: **[2013]**

(a) $\sin A\cos A+1$ (b) $\sec A\,\mathrm{cosec}A+1$

(c) $\tan A+\cot A$ (d) $\sec A+\mathrm{cosec}A$

ANSWER KEY

1. c **2.** a **3.** a **4.** b **5.** b

HINTS & SOLUTIONS

1.Sol: $8\cos x.\left[\left(\cos^2\frac{\pi}{6}-\sin^2 x\right)-\frac{1}{2}\right]=1$

$\Rightarrow \quad 8\cos x\left(\frac{3}{4}-\frac{1}{2}-1+\cos^2 x\right)=1$

$\Rightarrow \quad \frac{8\cos x}{4}\times\left(4\cos^2 x-1-2\right)=1$

$\Rightarrow \quad \cos 3x=4\cos^3 x-3\cos x$

$\Rightarrow \quad 2\times\cos 3x=1$

$\Rightarrow \quad \cos 3x=\frac{1}{2}$

i.e., $\quad 3x\in[0,3\pi]$

$\therefore \quad 3x=\frac{\pi}{3},2\pi-\frac{\pi}{3},2\pi+\frac{\pi}{3}$

$\therefore \quad \text{sum}=\frac{13\pi}{9}$

2.Sol: We have

$$5\tan^2 x - 5\cos^2 x = 2\left(2\cos^2 x - 1\right) + 9$$

$$\Rightarrow 5\tan^2 x - 5\cos^2 x = 4\cos^2 x - 2 + 9$$

$$\Rightarrow 5\tan^2 x = 9\cos^2 x + 7$$

$$\Rightarrow 5\left(\sec^2 x - 1\right) = 9\cos^2 x + 7$$

put $\cos^2 x = t$

$$\Rightarrow \frac{5}{t} - 9t - 12 = 0$$

$$\Rightarrow 9t^2 + 12t - 5 = 0$$

$$\Rightarrow (3t-1)(3t+5) = 0$$

$$\Rightarrow t = \frac{1}{3} \text{ as } t \neq -\frac{5}{3}.$$

$$\cos 2x = 2\cos^2 x - 1 = 2\left(\frac{1}{3}\right) - 1 = -\frac{1}{3}$$

$$\cos 4x = 2\cos^2 2x - 1 = 2\left(-\frac{1}{3}\right)^2 - 1 = -\frac{7}{9}$$

3.Sol: $\cos x + \cos 2x + \cos 3x + \cos 4x = 0$

$$\Rightarrow 2\cos 2x \cos x + 2\cos 2x \cos x = 0$$

$$\Rightarrow 2\cos x\left(2\cos\frac{5x}{2}\cos\frac{x}{2}\right) = 0$$

$$\cos x = 0, \cos\frac{5x}{2} = 0, \cos\frac{x}{2} = 0$$

$$x = \pi, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{5}, \frac{3\pi}{5}, \frac{7\pi}{5}, \frac{9\pi}{5}$$

$\therefore$ It has 7 solution

4.Sol: Let $f_k(x) = \frac{1}{k}\left(\sin^k + \cos^k x\right)$

Consider $f_4(x) - f_6(x) = \frac{1}{4}\left(\sin^4 x + \cos^4 x\right)$

$$-\frac{1}{6}\left(\sin^6 x + \cos^6 x\right)$$

$$= \frac{1}{4}\left[1 - 2\sin^2 x\cos^2 x\right] - \frac{1}{6}\left[1 - 3\sin^2 x\cos^2 x\right]$$

$$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

5.Sol: Given expression can be written as

$$\frac{\sin A}{\cos A} \times \frac{\sin A}{\sin A - \cos A} + \frac{\cos A}{\sin A} \times \frac{\cos A}{\cos A - \sin A}$$

$$= \frac{1}{\sin A - \cos A}\left\{\frac{\sin^3 A - \cos^3 A}{\cos A \sin A}\right\}$$

$$= \frac{\sin^2 A + \sin A\cos A + \cos^2 A}{\sin A \cos A}$$

$$= 1 + \sec A \operatorname{cosec} A$$