www.mtg.in | December 2016 | Pages 92 | ₹ 30

CONCEPT BOOSTERS Class XI | XII

CHALLENGING PROBLEMS

ACE YOUR WAY CBSE Class XI | XII

# MATHEMATICS today

India's #1 MATHEMATICS MONTHLY for JEE (Main & Advanced)

BRAIN @ WORK

WB JEE MOCK TEST PAPER SERIES 5

MATHS MUSING 10 GREAT PROBLEMS

MONTHLY PRACTICE PROBLEMS (XI & XII)

OLYMPIAD CORNER



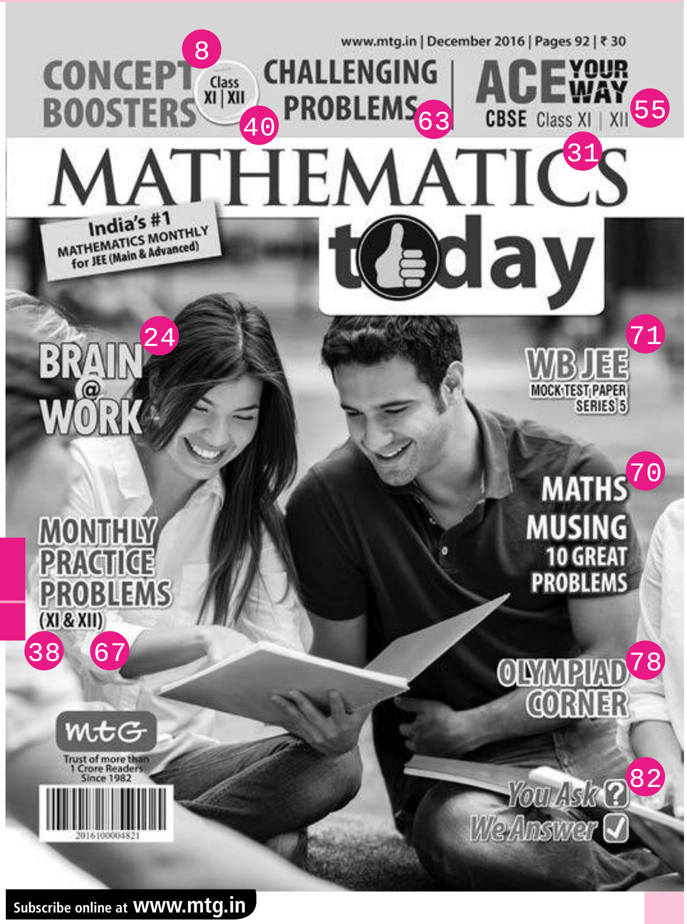
You Ask ? We Answer ✓

# MATHEMATICS today

Vol. XXXIV No. 12 December 2016

**Corporate Office:**
Plot 99, Sector 44 Institutional Area,
Gurgaon -122 003 (HR), Tel : 0124-6601200
e-mail : info@mtg.in website : www.mtg.in

**Regd. Office:**
406, Taj Apartment, Near Safdarjung Hospital,
Ring Road, New Delhi - 110029.
Managing Editor : Mahabir Singh
Editor : Anil Ahlawat


## CONTENTS



**Individual Subscription Rates**

| | 1 yr. | 2 yrs. | 3 yrs. |
|---|---|---|---|
| Mathematics Today | 330 | 600 | 775 |
| Chemistry Today | 330 | 600 | 775 |
| Physics For You | 330 | 600 | 775 |
| Biology Today | 330 | 600 | 775 |

**Combined Subscription Rates**

| | 1 yr. | 2 yrs. | 3 yrs. |
|---|---|---|---|
| PCM | 900 | 1500 | 1900 |
| PCB | 900 | 1500 | 1900 |
| PCMB | 1000 | 1800 | 2300 |

**Send D.D/M.O in favour of MTG Learning Media (P) Ltd.**
**Payments should be made directly to : MTG Learning Media (P) Ltd,**
**Plot 99, Sector 44 Institutional Area, Gurgaon - 122 003, Haryana.**
We have not appointed any subscription agent.


Owned, Printed and Published by Mahabir Singh from 406, Taj Apartment, New Delhi - 29 and printed by Personal Graphics and Advertisers (P) Ltd., Okhla Industrial Area, Phase-II, New Delhi. Readers are advised to make appropriate thorough enquiries before acting upon any advertisements published in this magazine. Focus/Infocus features are marketing incentives MTG does not vouch or subscribe to the claims and representations made by advertisers. All disputes are subject to Delhi jurisdiction only.
Editor : Anil Ahlawat

# CONCEPT BOOSTERS

Class XI

## Permutations and Combinations

This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

- **The Factorial :** Let $n$ be a positive integer. Then, the continued product of first $n$ natural numbers is called factorial $n$, to be denoted by $n\,!$ or $\underline{|n}$.
  Also, we define $0\,! = 1$.
  When $n$ is negative or a fraction, $n\,!$ is not defined.
  Thus, $n\,! = n\,(n-1)\,(n-2)\,......3{\cdot}2{\cdot}1$.
- **Exponent of prime $p$ in $n!$:** Let $p$ be a prime number and $n$ is a positive integer (*i.e.* natural number). If $E_p(n)$ denote the exponent of the prime $p$ in the positive integer $n$, then exponent of prime $p$ in $n!$ is denoted by $E_p(n!)$ and defined by
  $$E_p(n!) = E_p\,(1\,.\,2\,.\,3\,...\,(n-1)n)$$
  $$= \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + ..... + \left[\frac{n}{p^k}\right]$$
  where $k$ is the largest positive integer satisfying $p^k \le n < p^{k+1}$

### PERMUTATIONS

The ways of arranging a smaller or an equal number of persons or objects at a time from a given group of persons or objects with due regard being paid to the order of arrangement are called the (different) permutations.

The number of all Permutations of $n$ things taking $r$ at a time is denoted by ${}^np_r$. ${}^np_r$ is always a natural number and ${}^np_r = \dfrac{n!}{(n-r)!}$

Three different things $a$, $b$ and $c$ are given, then different arrangements which can be made by taking two things from three given things are $ab$, $ac$, $bc$, $ba$, $ca$, $cb$. Therefore the number of permutations will be 6.

### NUMBER OF PERMUTATIONS WITHOUT REPETITION

- Arranging $n$ objects, taken $r$ at a time equivalent to filling $r$ places from $n$ things.

| | | | | | | |
|---|---|---|---|---|---|---|
| $r$ places : | 1 | 2 | 3 | 4 | .... | $r$ |
| Number of choices : | $n$ | $(n-1)$ | $(n-2)$ | $(n-3)$ | .... | $n-(r-1)$ |

The number of ways of arranging
= The number of ways of filling $r$ places.
$= n(n-1)(n-2)\,....\,(n-r+1)$
$$= \frac{n(n-1)(n-2)....(n-r+1)((n-r)!)}{(n-r)!} = \frac{n!}{(n-r)!} = {}^nP_r$$

- The number of arrangements of $n$ different objects taken all at a time $= {}^nP_n = n!$
  - ${}^nP_0 = \dfrac{n!}{n!} = 1;\ {}^nP_r = n.{}^{n-1}P_{r-1}$
  - $0! = 1;\ \dfrac{1}{(-r)!} = 0$ or $(-r)! = \infty\ (r \in N)$

### NUMBER OF PERMUTATIONS WITH REPETITION

- The number of permutations (arrangements) of $n$ different objects, taken $r$ at a time, when each object may occur once, twice, thrice,........upto $r$ times in any arrangement = The number of ways of filling $r$ places where each place can be filled by any one of $n$ objects.

| | | | | | | |
|---|---|---|---|---|---|---|
| $r$ places : | 1 | 2 | 3 | 4 | .... | $r$ |
| Number of choices : | $n$ | $n$ | $n$ | $n$ | .... | $n$ |

The number of permutations = The number of ways of filling $r$ places = $(n)^r$.

- The number of arrangements that can be formed using $n$ objects out of which $p$ are identical (and of one kind) $q$ are identical (and of another kind), $r$ are identical (and of another kind) and the rest are distinct is $\dfrac{n!}{p!q!r!}$.

## CONDITIONAL PERMUTATIONS

- Number of permutations of $n$ dissimilar things taken $r$ at a time when $p$ particular things always occur $= {}^rP_p \cdot {}^{n-p}P_{r-p}$
- Number of permutations of $n$ dissimilar things taken $r$ at a time when $p$ particular things never occur $= {}^{n-p}P_r$
- The total number of permutations of $n$ different things taken not more than $r$ at a time, when each thing may be repeated any number of times, is $\dfrac{n(n^r-1)}{n-1}$.
- Number of permutations of $n$ different things, taken all at a time, when $m$ specified things always come together is $m! \times (n - m + 1)!$.
- Number of permutations of $n$ different things, taken all at a time, when $m$ specified things never come together is $n! - m! \times (n - m + 1)!$.
- Let there be $n$ objects, of which $m$ objects are alike of one kind, and the remaining $(n - m)$ objects are alike of another kind. Then, the total number of mutually distinguishable permutations that can be formed from these objects is $\dfrac{n!}{(m!)\times(n-m)!}$.

The above theorem can be extended further *i.e.*, if there are $n$ objects, of which $p_1$ are alike of one kind; $p_2$ are alike of $2^{nd}$ kind; $p_3$ are alike of $3^{rd}$ kind; ....; $p_r$ are alike of $r^{th}$ kind such that $p_1 + p_2 + ... p_r = n$; then the number of permutations of these $n$ objects is

$$\frac{n!}{(p_1!)\times(p_2!)\times......\times(p_r!)}.$$

## CIRCULAR PERMUTATIONS

In circular permutations, what really matters is the position of an object relative to the others.

Thus, in circular permutations, we fix the position of the one of the objects and then arrange the other objects in all possible ways.

There are two types of circular permutations :

- The circular permutations in which clockwise and the anticlockwise arrangements give rise to different permutations, *e.g.* Seating arrangements of persons round a table.
- The circular permutations in which clockwise and the anticlockwise arrangements give rise to same permutations, *e.g.* arranging some beads to form a necklace.

**Theorems on circular permutations**

- The number of circular permutations of $n$ different objects when clockwise and anticlockwise are taken as different is $(n - 1)!$.
- The number of ways in which $n$ different object when clockwise and anticlockwise are not not different is $\dfrac{1}{2}(n-1)!$.
- Number of circular permutations of $n$ different things, taken $r$ at a time, when clockwise and anticlockwise orders are taken as different is $\dfrac{{}^nP_r}{r}$.
- Number of circular permutations of $n$ different things, taken $r$ at a time, when clockwise and anticlockwise orders are not different is $\dfrac{{}^nP_r}{2r}$.

## COMBINATIONS

Each of the different groups or selections which can be formed by taking some or all of a number of objects, irrespective of their arrangements, is called combination.

- The number of all combinations of $n$ things, taken $r$ at a time is denoted by $C(n,r)$ or ${}^nC_r$ or $\dbinom{n}{r}$.

  ${}^nC_r$ is always a natural number and

  $${}^nC_r = \frac{n!}{r!(n-r)!}$$

**Difference between permutation and combination :**

- In a combination only selection is made whereas in a permutation not only a selection is made but also an arrangement in a definite order is considered.
- Each combination corresponds to many permutations. For example, the six permutations *ABC*, *ACB*, *BCA*, *BAC*, *CBA* and *CAB* correspond to the same combination *ABC*.

## NUMBER OF COMBINATIONS WITHOUT REPETITION

The number of combinations (selections or groups) that can be formed from $n$ different objects taken $r(0 \le r \le n)$ at a time is

$$^nC_r = \frac{n!}{r!(n-r)!}. \text{ Also } ^nC_r = {^nC_{n-r}}.$$

Let the total number of selections (or groups) = $x$. Each group contains $r$ objects, which can be arranged in $r$ ! ways. Hence the number of arrangements of $r$ objects $= x \times (r!)$. But the number of arrangements = $^nP_r$.

$$\Rightarrow x(r!) = {^nP_r} \Rightarrow x = \frac{^nP_r}{r!} \Rightarrow x = \frac{n!}{r!(n-r)!} = {^nC_r}$$

## NUMBER OF COMBINATIONS WITH REPETITION AND ALL POSSIBLE SELECTIONS

- The number of combinations of $n$ distinct objects taken $r$ at a time when any object may be repeated any number of times.
  = Coefficient of $x^r$ in $(1 + x + x^2 + .... + x^r)^n$
  = Coefficient of $x^r$ in $(1 - x)^{-n} = {^{n+r-1}C_r}$
- The total number of ways in which it is possible to form groups by taking some or all of $n$ things at a time is $^nC_1 + {^nC_2} + .... + {^nC_n} = 2^n - 1$.
- The total number of ways in which it is possible to make groups by taking some or all out of $n = (n_1 + n_2 + ...)$ things, when $n_1$ are alike of one kind, $n_2$ are alike of second kind, and so on is $\{(n_1 + 1)(n_2 + 1)...\} - 1$.
- The number of selections of $r$ objects out of $n$ identical objects is 1.
- Total number of selections of zero or more objects from $n$ identical objects is $n + 1$.
- The number of selections taking at least one out of $a_1 + a_2 + a_3 + .... + a_n + k$ objects, where $a_1$ are alike (of one kind), $a_2$ are alike (of second kind) and so on ... $a_n$ are alike (of $n^{th}$ kind) and $k$ are distinct
  $= [(a_1 + 1)(a_2 + 1)(a_3 + 1) ... (a_n + 1)]2^k - 1.$

## CONDITIONAL COMBINATIONS

- The number of ways in which $r$ objects can be selected from $n$ different objects if $k$ particular objects are
  - Always included $= {^{n-k}C_{r-k}}$
  - Never included $= {^{n-k}C_r}$
- The number of combinations of $n$ objects, of which $p$ are identical, taken $r$ at a time is
  $^{n-p}C_r + {^{n-p}C_{r-1}} + {^{n-p}C_{r-2}} + .... + {^{n-p}C_0}$,
  if $r \le p$ and
  $^{n-p}C_r + {^{n-p}C_{r-1}} + {^{n-p}C_{r-2}} + ... + {^{n-p}C_{r-p}}$,
  if $r > p$.

## DIVISION INTO GROUPS

**Case I**

- The number of ways in which $n$ different things can be arranged into $r$ different groups when order of the groups are considered is $^{n+r-1}P_n$ or $n!\ {^{n-1}C_{r-1}}$ according as blank group are or are not admissible.
- The number of ways in which $n$ different things can be distributed into $r$ different groups when order of groups are not considered is
  $r^n - {^rC_1}(r-1)^n + {^rC_2}(r-2)^n - .... + (-1)^{r-1}\,{^rC_{r-1}}$
  or coefficient of $x^n$ in $n!(e^x - 1)^r$.
  Here blank groups are not allowed.
- Number of ways in which $m \times n$ different objects can be distributed equally among $n$ persons (or numbered groups) = (number of ways of dividing into groups) × (number of groups) !
  $$= \frac{(mn)!n!}{(m!)^n n!} = \frac{(mn)!}{(m!)^n}.$$

**Case II**

- The number of ways in which $(m + n)$ different things can be divided into two groups which contain $m$ and $n$ things respectively is, $^{m+n}C_m \cdot {^nC_n} = \frac{(m+n)!}{m!n!}, m \ne n.$

  **Corollary:** If $m = n$, then the groups are equal size. Division of these groups can be given by two types.
  - **If order of group is not important :** The number of ways in which $2n$ different things can be divided equally into two groups is $\frac{(2n)!}{2!(n!)^2}$.
  - **If order of group is important :** The number of ways in which $2n$ different things can be divided equally into two distinct groups is $\frac{(2n)!}{2!(n!)^2} \times 2! = \frac{2n!}{(n!)^2}$.
- The number of ways in which $(m + n + p)$ different things can be divided into three groups which contain $m$, $n$ and $p$ things respectively is
  $$^{m+n+p}C_m \cdot {^{n+p}C_n} \cdot {^pC_p} = \frac{(m+n+p)!}{m!n!p!}, m \ne n \ne p.$$

**Corollary :** If $m = n = p$, then the groups are equal size. Division of these groups can be given by two types.

- **If order of group is not important :** The number of ways in which $3p$ different things can be divided equally into three groups is $\dfrac{(3p)!}{3!(p!)^3}$.
- **If order of group is important :** The number of ways in which $3p$ different things can be divided equally into three distinct groups is $\dfrac{(3p)!}{3!(p!)^3}3! = \dfrac{(3p)!}{(p!)^3}$.
- **If order of group is not important :** The number of ways in which $mn$ different things can be divided equally into $m$ groups is $\dfrac{mn!}{(n!)^m m!}$.
- **If order of group is important :** The number of ways in which $mn$ different things can be divided equally into $m$ distinct groups is $\dfrac{(mn)!}{(n!)^m m!} \times m! = \dfrac{(mn)!}{(n!)^m}$.

## DEARRANGEMENT

Any change in the given order of the things is called a de-arrangement.

If $n$ things form an arrangement in a row, the number of ways in which they can be de-arranged so that no one of them occupies its original place is $n!\left(1-\dfrac{1}{1!}+\dfrac{1}{2!}-\dfrac{1}{3!}+......+(-1)^n.\dfrac{1}{n!}\right)$.

## SOME IMPORTANT RESULTS

- Number of total different straight lines formed by joining the $n$ points on a plane of which $m$ ($< n$) are collinear is ${}^nC_2 - {}^mC_2 + 1$.
- Number of total triangles formed by joining the $n$ points on a plane of which $m$ ($< n$) are collinear is ${}^nC_3 - {}^mC_3$.
- Number of diagonals in a polygon of $n$ sides is ${}^nC_2 - n$.
- If $m$ parallel lines in a plane are intersected by a family of other $n$ parallel lines. Then total number of parallelograms so formed is ${}^mC_2 \times {}^nC_2$ *i.e.*, $\dfrac{mn(m-1)(n-1)}{4}$.
- Given $n$ points on the circumference of a circle, then
  - Number of straight lines = ${}^nC_2$
  - Number of triangles = ${}^nC_3$
  - Number of quadrilaterals = ${}^nC_4$.
- If $n$ straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent. Then the number of part into which these lines divide the plane is = $1 + \Sigma n$.

## MULTINOMIAL THEOREM

Let $x_1, x_2, ....., x_m$ be integers. Then number of solutions to the equation $x_1 + x_2 + .... + x_m = n$ ...(i)

Subject to the condition

$a_1 \le x_1 \le b_1,\ a_2 \le x_2 \le b_2,\ ....\ a_m \le x_m \le b_m$ ...(ii)

is equal to the coefficient of $x^n$ in

$$(x^{a_1}+x^{a_1+1}+...+x^{b_1})(x^{a_2}+x^{a_2+1}+...+x^{b_2})$$
$$...(x^{a_m}+x^{a_{m+1}}+...+x^{b_m}) \quad ...(iii)$$

This is because the number of ways, in which sum of $m$ integers in (i) equals $n$, is the same as the number of times $x^n$ comes in (iii).

- **Use of solution of linear equation and coefficient of a power in expansions to find the number of ways of distribution :** (i) The number of integral solutions of $x_1 + x_2 + x_3 + .... + x_r = n$ where $x_1 \ge 0$ $x_2 \ge 0, .... x_r \ge 0$ is the same as the number of ways to distribute $n$ identical things among $r$ persons.
  This is also equal to the coefficient of $x^n$ in the expansion of $(x^0 + x^1 + x^2 + x^3 + ...)^r$
  $= \text{coefficient of } x^n \text{ in } \left(\dfrac{1}{1-x}\right)^r$
  $= \text{coefficient of } x^n \text{ in } (1 - x)^{-r} = {}^{n+r-1}C_{r-1}$.
- The number of integral solutions of $x_1 + x_2 + x_3 + ... + x_r = n$ where $x_1 \ge 1, x_2 \ge 1, ... x_r \ge 1$ is same as the number of ways to distribute $n$ identical things among $r$ persons each getting at least 1. This also equal to the coefficient of $x^n$ in the expansion of $(x^1 + x^2 + x^3 + ....)^r$.
  $= \text{coefficient of } x^n \text{ in } \left(\dfrac{x}{1-x}\right)^r$
  $= \text{coefficient of } x^n \text{ in } x^r(1 - x)^{-r}$

## NUMBER OF DIVISORS

Let $N = p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}......p_k^{\alpha_k}$, where $p_1, p_2, p_3, ......p_k$ are different primes and $\alpha_1, \alpha_2, \alpha_3, ......, \alpha_k$ are natural numbers then :

- The total number of divisors of $N$ including 1 and $N$ is $= (\alpha_1+1)(\alpha_2+1)(\alpha_3+1)....(\alpha_k+1)$.

- The sum of these divisors is
$$=(p_1^0+p_1^1+p_1^2+...+p_1^{\alpha_1})(p_2^0+p_2^1+p_2^2+...+p_2^{\alpha_2})...$$
$$(p_k^0+p_k^1+p_k^2+....+p_k^{\alpha_k})$$
- The number of ways in which $N$ can be resolved as a product of two factors is
$$\begin{cases}\frac{1}{2}(\alpha_1+1)(\alpha_2+1)....(\alpha_k+1), \text{If } N \text{ is not a perfect square}\\ \frac{1}{2}[(\alpha_1+1)(\alpha_2+1).....(\alpha_k+1)+1], \text{If } N \text{ is a perfect square}\end{cases}$$
- The number of ways in which a composite number $N$ can be resolved into two factors which are relatively prime (or co-prime) to each other is equal to $2^{n-1}$ where $n$ is the number of different factors in $N$.

## SOME MORE TECHNIQUES

- ${}^nC_0 = {}^nC_n = 1$, ${}^nC_1 = n$.
- ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.
- ${}^nC_x = {}^nC_y \Leftrightarrow x = y$ or $x + y = n$.
- $n \cdot {}^{n-1}C_{r-1} = (n - r + 1)\, {}^nC_{r-1}$.
- If $n$ is even then the greatest value of ${}^nC_r$ is ${}^nC_{n/2}$.
- If $n$ is odd then the greatest value of ${}^nC_r$ is ${}^nC_{\frac{n+1}{2}}$ or ${}^nC_{\frac{n-1}{2}}$.
- ${}^nC_r = \frac{n}{r}\cdot{}^{n-1}C_{r-1}$.
- Number of selections of zero or more things out of $n$ different things is, ${}^nC_0 + {}^nC_1 + {}^nC_2 + .... + {}^nC_n = 2^n$.
- The number of ways of answering all of $n$ questions when each question has an alternative is $2^n$.
- ${}^nC_0 + {}^nC_2 + {}^nC_4 + ... = {}^nC_1 + {}^nC_3 + {}^nC_5 + ... = 2^{n-1}$.
- ${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + ... {}^{2n+1}C_n = 2^{2n}$.
- ${}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + {}^{n+3}C_n + ... + {}^{2n-1}C_n = {}^{2n}C_{n+1}$.
- Number of combinations of $n$ dissimilar things taken all at a time ${}^nC_n = \frac{n!}{n!(n-n)!} = \frac{1}{0!} = 1, \ (\because 0! = 1)$.
- **Gap method :** Suppose 5 males $A$, $B$, $C$, $D$, $E$ are arranged in a row as $\times A \times B \times C \times D \times E \times$. There will be six gaps between these five. Four in between and two at either end. Now if three females $P$, $Q$,$R$ are to be arranged so that no two are together we shall use gap method *i.e.*, arrange them in between these 6 gaps. Hence the answer will be ${}^6P_3$.
- **Together :** Suppose we have to arrange 5 persons in a row which can be done in 5 ! = 120 ways. But if two particular persons are to be together always, then we tie these two particular persons with a string. Thus we have 5 – 2 + 1 (1 corresponding to these two together) = 3 +1 = 4 units, which can be arranged in 4! ways. Now we loosen the string and these two particular can be arranged in 2! ways. Thus total arrangements = 24 × 2 = 48.
- Never together = Total – Together = 120 – 48 = 72.
- The number of ways in which $n$ (one type of different) things and $n$ (another type of different) things can be arranged in a row alternatively is $2 \cdot n! \cdot n!$.
- The number of ways in which $m$ things of one type and $n$ things of another type can be arranged in the form of a garland so that all the second type of things come together $= \frac{m!n!}{2}$ and no two things of second type come together $= \frac{(m-1)!\,{}^mP_n}{2}$.
- If we are given $n$ different digits ($a$, $a_2$, $a_3$ ... $a_n$) then sum of the digits in the unit place of all numbers formed without repetition is $(n - 1)!\ (a_1 + a_2 + a_3 + ... + a_n)$. Sum of the total numbers in this case can be obtained by applying the formula $(n - 1)!(a_1 + a_2 + a_3 + ... a_n)$. (1111 .... $n$ times).

## PROBLEMS

### Single Correct Answer Type

**1.** How many numbers can be formed from the digits 1, 2, 3, 4 when the repetition is not allowed

(a) ${}^4P_4$ (b) ${}^4P_3$
(c) ${}^4P_1 + {}^4P_2 + {}^4P_3$ (d) ${}^4P_1 + {}^4P_2 + {}^4P_3 + {}^4P_4$

**2.** How many even numbers of 3 different digits can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 (repetition is not allowed)

(a) 224 (b) 280
(c) 324 (d) None of these

**3.** The value of ${}^nP_r$ is equal to

(a) ${}^{n-1}P_r + r\,{}^{n-1}P_{r-1}$
(b) $n.\,{}^{n-1}P_r + {}^{n-1}P_{r-1}$
(c) $n({}^{n-1}P_r + {}^{n-1}P_{r-1})$
(d) ${}^{n-1}P_{r-1} + {}^{n-1}P_r$

**4.** Find the total number of 9 digit numbers which have all the digits different

(a) $9 \times 9!$ (b) $9!$
(c) $10!$ (d) None of these

**5.** The sum of the digits in the unit place of all numbers formed with the help of 3, 4, 5, 6 (without repetition) taken all at a time is

(a) 18 (b) 432 (c) 108 (d) 144

**6.** The figures 4, 5, 6, 7, 8 are written in every possible order. The number of numbers greater than 56000 is

(a) 72 (b) 96 (c) 90 (d) 98

**7.** The sum of all 4 digit numbers that can be formed by using the digits 2, 4, 6, 8 (repetition of digits is not allowed) is

(a) 133320 (b) 533280
(c) 53328 (d) None of these

**8.** The number of words which can be formed from the letters of the word MAXIMUM, if two consonants cannot occur together, is

(a) $4!$ (b) $3! \times 4!$
(c) $7!$ (d) None of these

**9.** In how many ways $n$ books can be arranged in a row so that two specified books are not together

(a) $n! - (n-2)!$ (b) $(n-1)!\,(n-2)$
(c) $n! - 2(n-1)$ (d) $(n-2)n!$

**10.** Numbers greater than 1000 but not greater than 4000 which can be formed with the digits 0, 1, 2, 3, 4 (repetition of digits is allowed), are

(a) 350 (b) 375 (c) 450 (d) 576

**11.** The number of numbers that can be formed with the help of the digits 1, 2, 3, 4, 3, 2, 1 so that odd digits always occupy odd places, is

(a) 24 (b) 18 (c) 12 (d) 30

**12.** In a circus there are ten cages for accommodating ten animals. Out of these four cages are so small that five out of 10 animals cannot enter into them. In how many ways will it be possible to accommodate ten animals in these ten cages

(a) 66400 (b) 86400
(c) 96400 (d) None of these

**13.** All possible four digit numbers are formed using the digits 0, 1, 2, 3 so that no number has repeated digits. The number of even numbers among them is

(a) 9 (b) 18
(c) 10 (d) None of these

**14.** The number of ways in which ten candidates $A_1 \cdot A_2, ..., A_{10}$ can be ranked such that $A_1$ is always above $A_{10}$ is

(a) $5!$ (b) $2(5!)$ (c) $10!$ (d) $\frac{1}{2}(10!)$

**15.** In how many ways can 5 boys and 5 girls stand in a row so that no two girls may be together

(a) $(5!)^2$ (b) $5! \times 4!$
(c) $5! \times 6!$ (d) $6 \times 5!$

**16.** How many numbers greater than hundred and divisible by 5 can be made from the digits 3, 4, 5, 6, if no digit is repeated

(a) 6 (b) 12 (c) 24 (d) 30

**17.** The number of 4 digit even numbers that can be formed using 0, 1, 2, 3, 4, 5, 6 without repetition is

(a) 120 (b) 300 (c) 420 (d) 20

**18.** Total number of four digit odd numbers that can be formed using 0, 1, 2, 3, 5, 7 (repitition is allowed) are

(a) 216 (b) 375 (c) 400 (d) 720

**19.** The number of words that can be formed out of the letters of the word ARTICLE so that the vowels occupy even places is

(a) 36 (b) 574 (c) 144 (d) 754

**20.** How many numbers lying between 999 and 10000 can be formed with the help of the digits 0,2,3,6,7,8 when the digits are not to be repeated

(a) 100 (b) 200 (c) 300 (d) 400

**21.** 20 persons are invited for a party. In how many different ways can they and the host be seated at a circular table, if the two particular persons are to be seated on either side of the host

(a) $20!$ (b) $2 \cdot 18!$
(c) $18!$ (d) None of these

**22.** The number of ways in which 5 beads of different colours form a necklace is

(a) 12 (b) 24 (c) 120 (d) 60

**23.** In how many ways 7 men and 7 women can be seated around a round table such that no two women can sit together

(a) $(7!)^2$ (b) $7! \times 6!$ (c) $(6!)^2$ (d) $7!$

**24.** If $\alpha = {}^mC_2$, then ${}^{\alpha}C_2$ is equal to

(a) ${}^{m+1}C_4$ (b) ${}^{m-1}C_4$
(c) $3 \cdot {}^{m+2}C_4$ (d) $3 \cdot {}^{m+1}C_4$

**25.** In a city no two persons have identical set of teeth and there is no person without a tooth. Also no person has more than 32 teeth. If we disregard the shape and size of tooth and consider only the positioning of the teeth, then the maximum population of the city is

(a) $2^{32}$ (b) $(32)^2 - 1$
(c) $2^{32} - 1$ (d) $2^{32-1}$

**26.** The least value of natural number $n$ satisfying $C(n, 5) + C(n, 6) > C(n + 1, 5)$ is

(a) 11 (b) 10
(c) 12 (d) 13

**27.** If ${}^{43}C_{r-6} = {}^{43}C_{3r+1}$, then the value of $r$ is

(a) 12 (b) 8 (c) 6 (d) 10

**28.** In an election there are 8 candidates, out of which 5 are to be choosen. If a voter may vote for any number of candidates but not greater than the number to be choosen, then in how many ways can a voter vote

(a) 216 (b) 114
(c) 218 (d) None of these

**29.** ${}^{n}C_r + {}^{n-1}C_r + \dots + {}^{r}C_r =$

(a) ${}^{n+1}C_r$ (b) ${}^{n+1}C_{r+1}$
(c) ${}^{n+2}C_r$ (d) $2^n$

**30.** In the 13 cricket players 4 are bowlers, then how many ways can form a cricket team of 11 players in which at least 2 bowlers included

(a) 55 (b) 72
(c) 78 (d) None of these

**31.** The number of groups that can be made from 5 different green balls, 4 different blue balls and 3 different red balls, if at least 1 green and 1 blue ball is to be included

(a) 3700 (b) 3720
(c) 4340 (d) None of these

**32.** In how many ways can 6 persons be selected from 4 officers and 8 constables, if at least one officer is to be included

(a) 224 (b) 672
(c) 896 (d) None of these

**33.** Out of 6 boys and 4 girls, a group of 7 is to be formed. In how many ways can this be done if the group is to have a majority of boys

(a) 120 (b) 90
(c) 100 (d) 80

**34.** The number of ways in which 10 persons can go in two boats so that there may be 5 on each boat, supposing that two particular persons will not go in the same boat is

(a) $\frac{1}{2}({}^{10}C_5)$ (b) $2({}^{8}C_4)$
(c) $\frac{1}{2}({}^{8}C_5)$ (d) None of these

**35.** The number of ways in which any four letters can be selected from the word 'CORGOO' is

(a) 15 (b) 11
(c) 7 (d) None of these

**36.** The total number of natural numbers of six digits that can be made with digits 1, 2, 3, 4, if all digits are to appear in the same number at least once, is

(a) 1560 (b) 840 (c) 1080 (d) 480

**37.** All possible two factors products are formed from numbers 1, 2, 3, 4, ...., 200. The number of factors out of the total obtained which are multiples of 5 is

(a) 5040 (b) 7180
(c) 8150 (d) None of these

**38.** The number of ways in which thirty five apples can be distributed among 3 boys so that each can have any number of apples, is

(a) 1332 (b) 666
(c) 333 (d) None of these

**39.** The number of ways in which four letters of the word 'MATHEMATICS' can be arranged is given by

(a) 136 (b) 192 (c) 1680 (d) 2454

**40.** A person is permitted to select at least one and at most $n$ coins from a collection of $(2n + 1)$ distinct coins. If the total number of ways in which he can select coins is 255, then $n$ equals

(a) 4 (b) 8 (c) 16 (d) 32

**41.** A student is to answer 10 out of 13 questions in an examination such that he must choose at least 4 from the first five question. The number of choices available to him is

(a) 140 (b) 196 (c) 280 (d) 346

## Multiple Correct Answer Type

**42.** Thirteen persons are sitting in a row. Number of ways in which four persons can be selected so that no two of them are consecutive is equal to _____

(a) number of ways in which all the letters of the word "MARRIAGE" are permutated if no two vowels are never together.

(b) number of numbers lying between 100 and 1000 using only the digits 1,2,3,4,5,6,7 without repetition.
(c) number of ways in which 4 alike chocolates can be distributed among 10 children so that each child getting at most one chocolate.
(d) number of triangles can be formed by joining 12 points in a plane, of which 5 are collinear

**43.** Suppose $A_1, A_2, ...., A_{20}$ are the vertices of a 20-sided regular polygon. Triangles with vertices among the vertices of the polygon are formed. Let $m$ be the number of non-isosceles (Scalene) triangles that can be formed one of whose sides is a side of the polygon and $n$ be the number of non-isosceles triangles that can be formed none of whose sides is a side of the polygon. Then

(a) $n = 2m$ (b) $m + n = 960$
(c) $m + n = 500$ (d) $n - m = 320$

**44.** Consider the set of all positive integers $n$ for which $f(n) = n! + (n + 1)! + (n + 2)!$ is divisible by 49.

(a) The number of integers $n$ in (1, 15) is 3
(b) The number of integers $n$ in (5, 17) is 4
(c) The number of integers $n$ in (1, 20) is 8
(d) The number of integers $n$ in (1, 20) is 9

**45.** Which of the following will not be true?

(a) The last two digits of $3^{100}$ will be 73
(b) The last two digits of $3^{50}$ will be 51
(c) The last two digits of $3^{50}$ will be 49
(d) The last three digits of $3^{50}$ will be 249

**46.** If $p, q, r, s, t$ be distinct primes and $N = pq^2r^3st$, then

(a) $N$ has 96 divisors
(b) $N$ can be written as a product of two positive integers in 96 ways
(c) $N$ can be written as a product of two positive integers in 48 ways
(d) $N$ can not be divisible by 13!

**47.** Triangles are formed by joining vertices of a octagon then number of triangles

(a) In which exactly one side common with the side of octagon is 32
(b) In which atmost one side common with the side of polygon is 48
(c) At least one side common with the side of polygon 50
(d) In total is 56

## Comprehension Type

**Paragraph for Q. No. 48 to 50**

Let $A, B, C, D, E$ be the smallest positive integers having 10, 12, 15, 16, 20 positive divisors respectively. Then

**48.** $A + B =$

(a) 108 (b) 110 (c) 126 (d) 130

**49.** $C + D =$

(a) 350 (b) 354 (c) 380 (d) 420

**50.** $A + E =$

(a) 288 (b) 320 (c) 350 (d) 380

**Paragraph for Q. No. 51 to 53**

Considering the rectangular hyperbola $xy = 15!$. The number of points $(\alpha, \beta)$ lying on it, where

**51.** $\alpha, \beta \in I$, is

(a) 2016 (b) 4032
(c) 4033 (d) 8064

**52.** $\alpha, \beta \in I^+$ and HCF $(\alpha, \beta) = 1$, is

(a) 64 (b) 785
(c) 4032 (d) 94185

**53.** $\alpha, \beta \in I^+$ and $\alpha$ divides $\beta$, is

(a) 96 (b) 511
(c) 1344 (d) 4032

**Paragraph for Q. No. 54 to 56**

Given are six 0's, five 1's and four 2's. Consider all possible permutations of all these numbers. [ A permutation can have its leading digit 0].

**54.** How many permutations have the first 0 preceding the first 1?

(a) ${}^{15}C_4 \times {}^{10}C_5$ (b) ${}^{15}C_5 \times {}^{10}C_4$
(c) ${}^{15}C_6 \times {}^{10}C_5$ (d) ${}^{15}C_4 \times {}^{9}C_4$

**55.** In how many permutations does the first 0 preceed the first 1 and the first 1 preceed first 2.

(a) ${}^{14}C_5 \times {}^{8}C_6$ (b) ${}^{14}C_5 \times {}^{8}C_4$
(c) ${}^{14}C_6 \times {}^{8}C_4$ (d) ${}^{12}C_5 \times {}^{7}C_4$

**56.** The no. of permutations in which all 2`s are together but no two of the zeroes are together is :

(a) 42 (b) 40
(c) 84 (d) 80

## Matrix-Match Type

**57.** Consider all possible permutations of the letters of the word ENDEANOEL. Match the Statements/ Expressions in Column I with the Statements/ Expressions in Column II.

| Column I | | Column II | |
|---|---|---|---|
| (A) | The number of permutations containing the word ENDEA, is | (p) | 5! |
| (B) | The number of permutations in which the letter E occurs in the first and the last positions, is | (q) | $2 \times 5!$ |
| (C) | The number of permutations in which none of the letters D, L, N occur in the last five positions, is | (r) | $7 \times 5!$ |
| (D) | The number of permutations in which the letters A, E, O occur only in odd positions, is | (s) | $21 \times 5!$ |

**58.** Match the following :

| Column I | | Column II | |
|---|---|---|---|
| (A) | The maximum number of points at which 5 straight lines intersect is | (p) | 120 |
| (B) | The number of distinct positive divisors of $2^4\ 3^5\ 5^3$ is | (q) | $2^n - 1$ |
| (C) | How many triangles can be drawn through 5 given points on a circle | (r) | ${}^5C_2$ |
| (D) | The value of $\sum_{r=1}^{n} \frac{{}^nP_r}{r!}$ | (s) | ${}^5C_3$ |

**59.** Match the following :

| Column I | | Column II | |
|---|---|---|---|
| (A) | The number of permutations of the letters of the word HINDUSTAN such that neither the pattern HIN nor DUS nor TAN appears, are | (p) | 169194 |
| (B) | Taking all the letters of the word MATHEMATICS how many wods can be formed in which either M or T are together? | (q) | $\frac{9.9!}{2!}$ |
| (C) | The number of ways in which we can choose 2 distinct integers from 1 to 100 such that difference between them is at most 10 is | (r) | ${}^{100}C_2 - {}^{90}C_2$ |
| (D) | The total number of eight-digit numbers, the sum of whose digits is odd, is | (s) | $45 \times 10^6$ |

## Integer Answer Type

**60.** Put numbers 1, 2, 3, 4, 5, 6, 7, 8 at the vertices of a cube, such that the sum of any three numbers on any face is not less than 10. The minimum sum of the four numbers on a face is $k$, then $k/2$ is equal to

**61.** The number of numbers from 1 to $10^6$ (both inclusive) in which two consecutive digits are same is equal to 402128 + $K$ where $K$ is a single digit number then $K$ must be equal to _______.

**62.** The number of polynomials of the form $x^3 + ax^2 + bx + c$ which are divisible by $x^2 + 1$ where $a, b, c \in \{1, 2, 3, ..., 10\}$ is $10K$, then $K$ is _______.

**63.** The number of ways of arranging 11 objects $A, B, C, D, E, F, \alpha, \alpha, \alpha, \beta, \beta$ so that every $\beta$ lie between two $\alpha$ (not necessarily adjacent) is $K \times 6! \times {}^{11}C_5$ , then $K$ is _____.

**64.** The number of positive integer solutions of $x + y + z = 10$, where $x, y, z$ are unequal is $(20 + K)$ then $K$ is

**65.** If number of numbers greater than 3000, which can be formed by using the digits 0, 1, 2, 3, 4, 5 without repetition, is $n$ then $\frac{n}{230}$ is equal to

## SOLUTIONS

**1. (d) :** Number of 1 digit numbers = ${}^4P_1$

Number of 2 digit numbers = ${}^4P_2$

Number of 3 digit numbers = ${}^4P_3$

Number of 4 digit numbers = ${}^4P_4$

Hence the required number of ways is sum of them.

**2. (a) :** The number will be even if last digit is 2, 4, 6 or 8 *i.e.*, the last digit can be filled in 4 ways and remaining two digits can be filled in ${}^8P_2$ ways. Hence required number of numbers are ${}^8P_2 \times 4 = 224$.

**3. (a) :** ${}^{n-1}P_r + r \cdot {}^{n-1}P_{r-1}$

$$= \frac{(n-1)!}{(n-1-r)!} + r\frac{(n-1)!}{(n-r)!} \qquad \left(\because\ {}^nP_r = \frac{n!}{(n-r)!}\right)$$

$$= \frac{(n-1)!}{(n-1-r)!}\left\{1 + r.\frac{1}{n-r}\right\}$$

$$= \frac{(n-1)!}{(n-1-r)!}\left(\frac{n}{n-r}\right) = \frac{n!}{(n-r)!} = {}^nP_r$$

**4. (a) :** There are 10 digits in all *viz.* 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The required 9 digit numbers = (Total number of 9 digit numbers including those numbers which have 0 at the first place) – (Total number of those

9 digit numbers which have 0 at the first place)

$$= {}^{10}P_9 - {}^{9}P_8 = \frac{10!}{1!} - \frac{9!}{1!} = 10! - 9! = (10-1)9! = 9\cdot 9!.$$

**5. (c) :** Required sum = 3!(3 + 4 + 5 + 6) = 6 × 18 = 108.

[If we fix 3 at the unit place, other three digits can be arranged in 3! ways similarly for 4, 5, 6.]

**6. (c) :** Required number of ways = 5! – 4! – 3! = 120 – 24 – 6 = 90.

[Number will be less than 56000 only if either 4 occurs on the first place or 5, 4 occurs on the first two places].

**7. (a) :** Sum of the digits in the unit place is 6(2 + 4 + 6 + 8) = 120 units. Similarly, sum of digits in tens place is 120 tens and in hundreds place is 120 hundreds etc. Sum of all the 24 numbers is $120(1 + 10 + 10^2 + 10^3) = 120 \times 1111 = 133320$.

**8. (a) :** • A • I • U •

The pointed places to be filled by MXMM .

Hence required number of ways $3! \times \frac{4!}{3!} = 4!$

{Since three vowels can be arranged in 3! ways also}.

**9. (b) :** Total number of arrangements of $n$ books = $n!$.

If two specified books always together then number of ways = $(n - 1)! \times 2$

Hence required number of ways = $n! - (n - 1)! \times 2$

$= n(n - 1)! - (n - 1)! \times 2 = (n - 1)!\,(n - 2)$.

**10. (b) :** Numbers greater than 1000 and less than or equal to 4000 will be of 4 digits and will have either 1 (except 1000) or 2 or 3 in the first place.

After fixing 1st place, the second place can be filled by any of the 5 numbers. Similarly third place can be filled up in 5 ways and 4th place can be filled up in 5 ways. Thus there will be 5 × 5 × 5 = 125 ways in which 1 will be in first place but this include 1000 also hence there will be 124 numbers having 1 in the first place. Similarly 125 for each 2 or 3. One number will be in which 4 in the first place *i.e.* 4000. Hence the required numbers are 124 + 125 + 125 + 1 = 375.

**11. (b) :** The 4 odd digits 1, 3, 3, 1 can be arranged in the 4 odd places in $\frac{4!}{2!2!} = 6$ ways and 3 even digits 2, 4, 2 can be arranged in the three even places in $\frac{3!}{2!} = 3$ ways. Hence the required number of ways = 6 × 3 = 18.

**12. (b) :** At first we have to accommodate those 5 animals in cages which can not enter in 4 small cages, therefore number of ways are ${}^6P_5$. Now after accommodating 5 animals we left with 5 cages and 5 animals, therefore number of ways are 5!. Hence required number of ways = ${}^6P_5 \times 5! = 86400$.

**13. (c) :** In forming even numbers, the position on the right can be filled either 0 or 2. When 0 is filled, the remaining positions can be filled in 3! ways and when 2 is filled, the position on the left can be filled in 2 ways (0 cannot be used) and the middle two positions in 2! ways (0 can be used). Therefore the number of even numbers formed = 3! + 2(2!) = 10.

**14. (d) :** Without any restriction the 10 persons can be ranked among themselves in 10! ways; but the number of ways in which $A_1$ is above $A_{10}$ and the number of ways in which $A_{10}$ is above $A_1$ make up 10!. Also the number of ways in which $A_1$ is above $A_{10}$ is exactly same as the number of ways in which $A_{10}$ is above $A_1$.

Therefore the required number of ways $= \frac{1}{2}(10!)$.

**15. (c) :** 5 boys can be stand in a row 5! ways

Now, two girls can't stand in a row together in ${}^6P_5$ ways.

Total no. of required arrangement = $5! \times {}^6P_5 = 5! \times 6!$.

**16. (b) :** Numbers which are divisible by 5 have '5' fixed in extreme right place

| 3 Digit Numbers | | | 4 Digit Numbers | | | |
|---|---|---|---|---|---|---|
| H | T | U | Th | H | T | U |
| × | × | 5 | × | × | × | 5 |
| ${}^3P_2$ ways | | | ${}^3P_3$ ways | | | |
| $=\frac{3!}{1!}=3\times 2$ | | | $=\frac{3!}{0!}=3\times 2$ | | | |

⇒ Total ways =12.

**17. (c) :** The units place can be filled in 4 ways as any one of 0, 2, 4 or 6 can be placed there. The remaining three places can be filled in with remaining 6 digits in ${}^6P_3$ = 120 ways. So, total number of ways = 4 × 120 = 480. But, this includes those numbers in which 0 is fixed in extreme left place. Numbers of such numbers $= 3 \times {}^5P_2 = 3 \times 5 \times 4 = 60$

∴ Required number of ways = 480 – 60 = 420.

**18. (d) :** 0, 1, 2, 3, 5, 7 : Six digits

The last place can be filled in by 1, 3, 5, 7. *i.e.*, 4 ways as the number is to be odd. We have to fill in the remaining 3 places of the 4 digit number. Since repetition is allowed each place can be filled in 6 ways. Hence the 3 place can be filled in 6 × 6 × 6 = 216 ways.

But in case of 0 = 216 – 36 = 180 ways.

Hence by fundamental theorem, the total number will be = 180 × 4 = 720.

**19. (c) :** Out of 7 places, 4 places are odd and 3 even. Therefore 3 vowels can be arranged in 3 even places in ${}^3P_3$ ways and remaining 4 consonants can be arranged in 4 odd places in ${}^4P_4$ ways.

Hence required no. of ways $= {}^3P_3 \times {}^4P_4 = 144$.

**20. (c) :** The numbers between 999 and 10000 are of four digit numbers.

The four digit numbers formed by digits 0, 2,3,6,7,8 are ${}^6P_4 = 360$.

But here those numbers are also involved which begin from 0. So we take those numbers as three digit numbers.

Taking initial digit 0, the number of ways to fill remaining 3 places from five digits 2,3,6,7,8 are ${}^5P_3 = 60$

So the required numbers = 360 – 60 = 300.

**21. (b) :** There are 20 + 1 = 21 persons in all. The two particular persons and the host be taken as one unit so that these remain 21 – 3 + 1 = 19 persons to be arranged in 18! ways. But the two person on either side of the host can themselves be arranged in 2! ways. Hence there are 2! 18! ways or 2 · 18! ways.

**22. (a) :** The number of ways in which 5 beads of different colours can be arranged in a circle to form a necklace are (5 – 1)! = 4!.

But the clockwise and anticlockwise arrangement are not different (because when the necklace is turned over one gives rise to another)

Hence the total number of ways of arranging the beads

$$=\frac{1}{2}(4!)=12.$$

**23. (b) :** Fix up 1 man and the remaining 6 men can be seated in 6! ways. Now no two women are to sit together and as such the 7 women are to be arranged in seven empty seats between two consecutive men and number of arrangement will be 7!. Hence by fundamental theorem the total number of ways = 7! × 6!.

**24. (d) :** $\alpha = {}^mC_2 \Rightarrow \alpha = \dfrac{m(m-1)}{2}$

$$\therefore\ {}^{\alpha}C_2 = {}^{m(m-1)/2}C_2 = \frac{1}{2}\cdot\frac{m(m-1)}{2}\left\{\frac{m(m-1)}{2}-1\right\}$$

$$=\frac{1}{8}m(m-1)(m-2)(m+1)$$

$$=\frac{1}{8}(m+1)\,m(m-1)(m-2)=3\,.\,{}^{m+1}C_4$$

**25. (c) :** We have 32 places for teeth. For each place we have two choices either there is a tooth or there is no tooth. Therefore the number of ways to fill up these places is $2^{32}$ . As there is no person without a tooth, the maximum population is $2^{32} - 1$.

**26. (a) :** ${}^nC_5 + {}^nC_6 > {}^{n+1}C_5 \Rightarrow {}^{n+1}C_6 > {}^{n+1}C_5$

$$\Rightarrow\ \frac{(n+1)!}{6!(n-5)!}\cdot\frac{5!(n-4)!}{(n+1)!}>1\Rightarrow\frac{(n-4)}{6}>1$$

$\Rightarrow$ $n - 4 > 6 \Rightarrow n > 10$

Hence, least value of $n = 11$.

**27. (a) :** Given, ${}^{43}C_{r-6} = {}^{43}C_{3r+1}$

$\Rightarrow$ $r - 6 = 3r + 1$ or $r - 6 + 3r + 1 = 43$

$$\Rightarrow\ r=-\frac{7}{2}\text{(impossible) or } r=12.$$

**28. (c) :** Required number of ways

$= {}^8C_1 + {}^8C_2 + {}^8C_3 + {}^8C_4 + {}^8C_5$

= 8 + 28 + 56 + 70 + 56 = 218

{Since voter may vote to one, two, three, four or all candidates}.

**29. (b) :** ${}^rC_r + {}^{r+1}C_r + {}^{r+2}C_r \ldots + {}^{n-1}C_r + {}^nC_r$

$= {}^{r+1}C_{r+1} + {}^{r+1}C_r + {}^{r+2}C_r + \ldots + {}^{n-1}C_r + {}^nC_r$

$= {}^{r+2}C_{r+1} + {}^{r+2}C_r + \ldots + {}^{n-1}C_r + {}^nC_r$

$= {}^{r+3}C_{r+1} + \ldots + {}^{n-1}C_r + {}^nC_r$.

On solving similar way, we get

${}^{n-1}C_{r+1} + {}^{n-1}C_r + {}^nC_r = {}^nC_{r+1} + {}^nC_r = {}^{n+1}C_{r+1}$.

**30. (c) :** The number of ways can be given as follows

2 bowlers and 9 other players = ${}^4C_2 \times {}^9C_9$

3 bowlers and 8 other players = ${}^4C_3 \times {}^9C_8$

4 bowlers and 7 other players = ${}^4C_4 \times {}^9C_7$

Hence required number of ways

= 6 × 1 + 4 × 9 + 1 × 36 = 78.

**31. (b) :** At least one green ball can be selected out of 5 green balls in $2^5 - 1$ *i.e.*, in 31 ways. Similarly at least one blue ball can be selected from 4 blue balls in $2^4 - 1 = 15$ ways. And at least one red or not red can be select in $2^3 = 8$ ways.

Hence required number of ways = 31 × 15 × 8 = 3720.

**32. (c) :** Required number of ways
$= {}^4C_1 \times {}^8C_5 + {}^4C_2 \times {}^8C_4 + {}^4C_3 \times {}^8C_3 + {}^4C_4 \times {}^8C_2$
$= 4 \times 56 + 6 \times 70 + 4 \times 56 + 1 \times 28 = 896.$

**33. (c) :** 1 girl and 6 boys $= {}^4C_1 \times {}^6C_6 = 4$
2 girls and 5 boys $= {}^4C_2 \times {}^6C_5 = 36$
3 girls and 4 boys $= {}^4C_3 \times {}^6C_4 = 60$
Hence total ways $= 60 + 36 + 4 = 100.$

**34. (b) :** First omit two particular persons, remaining 8 persons may be 4 in each boat. This can be done in ${}^8C_4$ways. The two particular persons may be placed in two ways one in each boat. Therefore total number of ways $= 2 \times {}^8C_4$.

**35. (c) :** Four letters can be selected in the following ways
(i) All different *i.e.* C, O, R, G.
(ii) 2 like and 2 different.
(iii) 3 like and 1 different *i.e.* three O and 1 from R, G and C.
The number of ways in (i) is ${}^4C_4 = 1$
The number of ways in (ii) is $1 \cdot {}^3C_2 = 3$
The number of ways in (iii) is $1 \times {}^3C_1 = 3$
Therefore, required number of ways $= 1 + 3 + 3 = 7.$

**36. (a) :** There can be two types of numbers :
(i) Any one of the digits 1, 2, 3, 4 repeats thrice and the remaining digits only once *i.e.* of the type 1, 2, 3, 4, 4, 4 etc.
(ii) Any two of the digits 1, 2, 3, 4 repeat twice and the remaining two only once *i.e.* of the type 1, 2, 3, 3, 4, 4 etc.
Now number of numbers of the (i) type
$= \frac{6!}{3!} \times {}^4C_1 = 480$
Number of numbers of the (ii) type
$= \frac{6!}{2!2!} \times {}^4C_2 = 1080$
Therefore the required number of numbers
$= 480 + 1080 = 1560.$

**37. (b) :** The total number of two factor products $= {}^{200}C_2$. The number of numbers from 1 to 200 which are not multiples of 5 is 160. Therefore total number of two factor products which are not multiple of 5 is ${}^{160}C_2$. Hence the required number of factors
$= {}^{200}C_2 - {}^{160}C_2 = 7180.$

**38. (b) :** The required number
= Coefficient of $x^{35}$ in $(1 + x + x^2 + .... + x^{35})^3$
$= {}^{3+35-1}C_{3-1} = {}^{37}C_2 = 666$

**39. (d) :** Word 'MATHEMATICS' has 2M, 2T, 2A, H, E, I, C, S. Therefore 4 letters can be chosen in the following ways.
**Case I :** 2 alike of one kind and 2 alike of second kind
*i.e.*, ${}^3C_2 \Rightarrow$ No. of words $= {}^3C_2 \frac{4!}{2!2!} = 18$
**Case II :** 2 alike of one kind and 2 different
*i.e.*, ${}^3C_1 \times {}^7C_2 \Rightarrow$ No. of words $= {}^3C_1 \times {}^7C_2 \times \frac{4!}{2!} = 756$
**Case III :** All are different
*i.e.*, ${}^8C_4 \Rightarrow$ No. of words $= {}^8C_4 \times 4! = 1680.$
Hence total number of words are 2454.

**40. (a) :** Since the person is allowed to select at most $n$ coins out of $(2n + 1)$ coins, therefore in order to select one, two, three, ...., $n$ coins. Thus, if $T$ is the total number of ways of selecting coins, then
$T = {}^{2n+1}C_1 + {}^{2n+1}C_2 + ... + {}^{2n+1}C_n = 255$ ...(i)
Again the sum of binomial coefficients is
${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + ... + {}^{2n+1}C_n + {}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + ... + {}^{2n+1}C_{2n+1} = (1+1)^{2n+1} = 2^{2n+1}$
$\Rightarrow {}^{2n+1}C_0 + 2({}^{2n+1}C_1 + {}^{2n+1}C_2 + ... + {}^{2n+1}C_n) + {}^{2n+1}C_{2n+1} = 2^{2n+1}$
$\Rightarrow 1 + 2(T) + 1 = 2^{2n+1} \Rightarrow 1 + T = \frac{2^{2n+1}}{2} = 2^{2n}$
$\Rightarrow 1 + 255 = 2^{2n} \Rightarrow 2^{2n} = 2^8 \Rightarrow n = 4$

**41. (b) :** As for given question two cases are possible.
(i) Selecting 4 out of first 5 questions and 6 out of remaining 8 questions $= {}^5C_4 \times {}^8C_6 = 140$ choices.
(ii) Selecting 5 out of first 5 questions and 5 out of remaining 8 questions $= {}^5C_5 \times {}^8C_5 = 56$ choices.
$\therefore$ Total no. of choices $= 140 + 56 = 196.$

**42. (b, c, d) :** $x_1 + x_2 + x_3 + x_4 + x_5 = 9$, $x_1, x_5 \geq 0$
$x_2, x_3, x_4 \geq 1$, number of solutions are 210
(a) $5 \times 12 \times 12 = 720$ (b) ${}^7P_3 = 210$
(c) ${}^{10}C_4 = 210$ (d) ${}^{12}C_3 - {}^5C_3 = 210$

**43. (a, b, d) :** Number of isosceles triangles
$= 20 \times 9 = 180$
$m = 20 \times 16 = 320$
$n = {}^{20}C_3 - (180 + 320) = 640$

**44. (a, b, d) :** $f(n) = (n!)\,(1 + n + 1 + (n + 1)(n + 2))$
$= (n + 2)^2(n!)$
$49 / f(n) \Rightarrow 7 / n + 2$ or $49 / n!$

**45. (a, b) :** $3^{100} = 9^{50} = (10 - 1)^{50}$
$= 10^{50} - {}^{50}C_1 10^{49} + ... - {}^{50}C_{49} 10 + {}^{50}C_{50}$
= a multiple of 100 + 1
$3^{50} = (10 - 1)^{25} = 10^{25} - {}^{25}C_1(10)^{24} + ... + {}^{25}C_{24}10 - {}^{25}C_{25}$
= a multiple of 100 + 249

**46. (a, c, d) :** No. of divisors = (1 + 1)(2 + 1)(3 + 1)(1 + 1)(1 + 1) = 96

Since there are 6 primes which are $\le 13$ and $N$ contain only five distinct primes, $N$ can not be divisible By 13!.

**47. (a, b, d) :** Total number of triangles = ${}^8C_3$ = 56

Number of triangles having exactly one side common with the polygon = 8 × 4 = 32

Number of triangles having exactly two side common with the polygon = 8

Number of triangles having no side common with the polygon = 16

**(48 - 50) :**

$10 = 2 \cdot 5 \Rightarrow A = 2^4 \cdot 3 = 48$

$12 = 2 \cdot 2 \cdot 3 \Rightarrow B = 2^2 \cdot 3 \cdot 5 = 60$

$15 = 3 \cdot 5 \Rightarrow C = 2^4 \cdot 3^2 = 144$

$16 = 2 \cdot 2 \cdot 2 \cdot 2 \Rightarrow D = 2 \cdot 3 \cdot 5 \cdot 7 = 210$

$20 = 2 \cdot 2 \cdot 5 \Rightarrow E = 2^4 \cdot 3 \cdot 5 = 240$

**48. (a) :** $A + B = 48 + 60 = 108$

**49. (b) :** $C + D = 144 + 210 = 354$

**50. (a) :** $A + E = 48 + 240 = 288$

**51. (d) :** $xy = 15! = 2^{11}3^6\ 5^3\ 7^2\ 11^1\ 13^1$

Number of positive integral solutions = no. of ways of fixing $x$ = the number of factors of 15!

= (1 + 11) (1 + 6) (1 + 3) (1 + 2) (1 + 1) (1 + 1) = 4032

Total number of integral solutions (positive or negative) = 2 × 4032 = 8064

**52. (a) :** HCF (α, β) = 1. α and β will not have common factor other than 1 so, identical prime numbers should not be separated. *e.g.* $2^{11}$ will completely go with either α or β.

So the number of solutions = 2 × 2 × 2 × 2 × 2 × 2 = 64.

**53. (a) :** The largest number whose perfect square can be made with 15! is $2^5\ 3^3\ 5^1\ 7^1$

So that number of ways of selecting $x$ will be (1 + 5) (1 + 3) (1 + 1) (1 + 1) = 96.

**54. (a) :** The number of ways of arranging 2's is ${}^{15}C_4$. Fill the first empty position left after arranging the 2's with a 0(1 way) and pick the remaining five places the position the remaining five zeros $\rightarrow {}^{10}C_5$ ways.

$\therefore\ {}^{15}C_4 \times 1 \times {}^{10}C_5$

**55. (b) :** Put 0 in the first position, ( 1 way). Pick five other positions for the remaining 0's (${}^{14}C_5$ ways), put a 1 in the first of the remaining positions (1 way), then arrange the remaining four 1's (${}^8C_4$ ways)

$\therefore\ {}^{14}C_5 \times {}^8C_4$

**56. (a)**

**57. A → p; B → s; C → q; D → q**

ENDEANOEL

(A) Consider ENDEA as a single unit

$\Rightarrow$ ENDEA,N,O,E,L $\Rightarrow 5!$

(B) After filling E's at first and last positions remaining letters are N, D, A, N, O, E, L $\Rightarrow \dfrac{\underline{|7}}{\underline{|2}} = 21 \times 5!$

(C) D, L, N, N can't be present in the last 5 positions $\Rightarrow$ They occupy 1st four positions, for which number of ways $= \dfrac{4!}{2!} = 12$

And the remaining 5 letters : E, E, E, A, O will occupy last 5 positions in $\dfrac{5!}{3!}$ ways

$\Rightarrow$ Required no. of ways $= 12 \times \dfrac{5!}{3!} = 2 \times 5!$

(D) A,E,O $\Rightarrow$ A,E,E,E,O

In fact there are only 5 odd positions available.

**58. A → r; B → p; C → s; D → q**

(A) Two straight lines intersect at only one point. For selecting two out of 5 straight lines is ${}^5C_2$. So maximum number of point of intersection is ${}^5C_2$.

(B) The number of distinct positive divisors of $2^4\ 3^5\ 5^3 = (4 + 1)\ (5 + 1)\ (3 + 1) = 120$

(C) Total number of triangles formed = ${}^5C_3$

(D) $\displaystyle\sum_{r=1}^{n} \frac{{}^nP_r}{r!} = {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n - 1$

**59. A → p; B → q; C → r; D → s**

(A) Total number of permutations $= \dfrac{9!}{2!}$

Number of those containing HIN = 7!

Number of those containing DUS $= \dfrac{7!}{2!}$

Number of those containing TAN = 7!

Number of those containing HIN and DUS = 5!

Number of those containing HIN and TAN = 5!

Number of those containing TAN and DUS = 5!

Number of those containing HIN, DUS and TAN = 3!

Required numbers

$$= \frac{9!}{2!} - \left(7! + 7! + \frac{7!}{2}\right) + 3 \times 5! - 3! = 169194$$

**Solution Sender of Maths Musing**

**SET-167**

1. N. Jayanthi Hyderabad

(B) M, M, T, T, A, A, H, E, I, C, S

(Number of words in which both M are together) + (Number of words in which both T are together).
–(Number of words in which both T and both M are together) = required number of words

Required number of words

$$=\frac{10!}{2!2!}+\frac{10!}{2!2!}-\frac{9!}{2!}=\frac{5.9!+5.9!-9!}{2!}=\frac{9.9!}{2!}$$

(C) Let the chosen integers be $x_1$ and $x_2$

Let there be '$a$' integer before $x_1$, '$b$' integer between $x_1$ and $x_2$ and '$c$' integer after $x_2$

$\therefore$ $a + b + c = 98$. Where $a \geq 0$, $b < 10$, $c \geq 0$

Now if we consider the choices where difference is at least 11, then the number of solutions is $^{88+3-1}C_{3-1} = {}^{90}C_2$

$\therefore$ Number of ways in which $b$ is less than 10 is $^{100}C_2 - {}^{90}C_2$

(D) The numbers will vary from 10000000 to 99999999. If sum of digits of a particular number is even, then the sum of digits of its next consecutive number will be odd.

As sum of digits of first number is odd and sum of digits of last number is even.

So number of numbers with sum of digits as odd

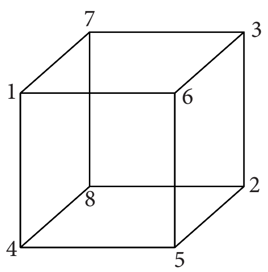
$$=\frac{\text{total number of 8} - \text{digit numbers}}{2}$$

$$=\frac{90000000}{2}=45\times10^6$$

**60. (8) :** Suppose that the four numbers on face of the cube is $a_1$, $a_2$, $a_3$, $a_4$ such that their sum reaches the minimum and $a_1 < a_2 < a_3 < a_4$.

Since the maximum sum of any three numbers less than 5 is 9, we have $a_4 \geq 6$ and $a_1 + a_2 + a_3 + a_4 \geq 16$.

As seen in figure, we have $2 + 3 + 5 + 6 = 16$ and that means minimum sum of four numbers on a face is 16.




**61. (2) :** No. of $n$ digit numbers in which no two consecutive digits are same $= 9^n$

$\Rightarrow$ no. of numbers from 1 to $10^6$ in which no two consecutive digits are same $= \sum_{n=1}^{6} 9^n = 597870$

Required numbers $= 10^6 - 597870 = 402130$
$= 402128 + 2$

$\therefore$ $K = 2$

**62. (1) :** $x^2 + 1 = (x + i)(x - i)$

$b = 1$, $a = c$

No.of ways of choosing $a$, $b$, $c = 10 = 10 \times 1$

$\therefore$ $K = 1$

**63. (3) :** There are three major ways $\alpha\alpha\beta\beta\alpha$, $\alpha\beta\beta\alpha\alpha$ and $\alpha\beta\alpha\beta\alpha$

Each major way has six empty spaces. The number of ways of putting letters at these empty spaces must be non-negative integer function of $x_1 + x_2 + \ldots + x_6 = 6 = {}^{6+6-1}C_{6-1} = {}^{11}C_5$

No.of arrangements is $= 3 \times {}^{11}C_5 \times 6! \Rightarrow K = 3$

**64. (4) :** $x < y < z$, these are 1 2 7, 1 3 6, 1 4 5, 2 3 5

total $= 3! \times 4 = 24 = 20 + 4$

$\therefore$ $K = 4$

**65. (6) :** No. of 4 digit numbers $= 3 \times 5 \times 4 \times 3 = 180$

No. of 5 digit numbers $= 5 \times 5 \times 4 \times 3 \times 2 = 600$

No. of 6 digit numbers $= 5 \times 5 \times 4 \times 3 \times 2 \times 1 = 600$

$n = 1380$

$$\Rightarrow \frac{n}{230}=6$$

# BRAIN @ WORK

## CIRCLES

This article is a collection of shortcut methods, important formulas and MCQs along with their detailed solutions which provides an extra edge to the readers who are preparing for various competitive exams like JEE(Main & Advanced) and other PETs.

### DEFINITION

A circle is the locus of a point which moves in a plane such that its distance from a fixed point in the plane is always a constant. The fixed point is called the centre and the constant distance is called the radius of the circle.

### EQUATION OF A CIRCLE PASSING THROUGH THREE NON-COLLINEAR POINTS

Let the equation of the circle passing through three non-collinear points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be

$x^2 + y^2 + 2gx + 2fy + c = 0$ ...(1)

If these three points lie on the circle (1), then their co-ordinates must satisfy its equation. Hence,

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \qquad ...(2)$$

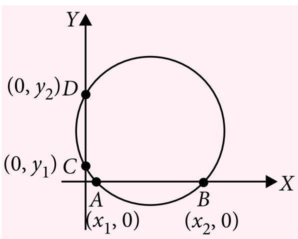
$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \qquad ...(3)$$

$$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0 \qquad ...(4)$$

$g, f, c$ are obtained from (2), (3) and (4). Then to find the circle (1). Substitute the value of $g, f, c$ so obtained in equation (1).

### INTERCEPTS MADE ON THE AXIS BY A CIRCLE

- The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ intersects the $x$-axis at $A(x_1, 0)$ and $B(x_2, 0)$ then $AB = |x_1 - x_2|$ $= 2\sqrt{(g^2 - c)}$
- The circle intercepts the $y$-axis at $C(0, y_1)$ and $D(0, y_2)$, then $CD = |y_1 - y_2|$ $= 2\sqrt{(f^2 - c)}$




**Remarks :**

(i) Length of intercepts are always positive.

(ii) If circle touches $x$-axis then $AB = 0 \Rightarrow c = g^2$ and if circle touches $y$-axis then $CD = 0 \Rightarrow c = f^2$

(iii) If circle touches both axes then $AB = CD = 0$ $\Rightarrow c = g^2 = f^2$

### EQUATIONS OF A CIRCLE IN DIFFERENT CONDITIONS

- When the circle passes through the origin (0, 0) and has intercepts $2\alpha$ and $2\beta$ on the $x$-axis and $y$-axis respectively. Centre $(\alpha, \beta)$, radius $= \sqrt{\alpha^2 + \beta^2}$
  $\therefore$ Equation of the circle is $(x - \alpha)^2 + (y - \beta)^2 = \alpha^2 + \beta^2$.
- When the circle touches $x$-axis and having centre at $(\alpha, \beta)$, radius $= \beta$.
  $\therefore$ Equation of circle is $(x - \alpha)^2 + (y - \beta)^2 = \beta^2$
- When the circle touches $y$-axis and having centre at $(\alpha, \beta)$, radius $= \alpha$.
  $\therefore$ Equation of circle is $(x - \alpha)^2 + (y - \beta)^2 = \alpha^2$
- When the circle touches both axes and having centre $(\alpha, \alpha)$, radius $= \alpha$.
  $\therefore$ Equation of circle is $(x - \alpha)^2 + (y - \alpha)^2 = \alpha^2$
- When the circle passes through the origin and centre lies on $x$-axis. *i.e.*, centre $(\alpha, 0)$, radius $= \alpha$
  $\therefore$ Equation of circle is $(x - \alpha)^2 + y^2 = \alpha^2$
- When the circle passes through the origin and centre lies on $y$-axis. *i.e.*, centre $(0, \alpha)$, radius $= \alpha$
  $\therefore$ Equation of circle is $x^2 + (y - \alpha)^2 = \alpha^2$

### POSITION OF A POINT AND ITS MAXIMUM AND MINIMUM DISTANCE W.R.T. A CIRCLE

A point $P(x_1, y_1)$ may lies outside, on or inside a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$

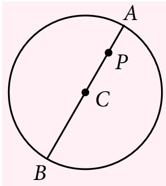
Sanjay Singh Mathematics Classes, Chandigarh, Ph : 9888228231, 9216338231

**Case I :** If $P$ lies inside the circle

$P$ lies inside if $PC < r$

$\Rightarrow \quad x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0$

The minimum distance of $P$ from circle $= PA = CA - CP = |r - CP|$.




and the maximum distance of $P$ from circle $= PB = CB + CP = |r + CP|$.

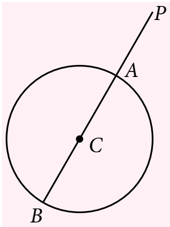
**Case II :** If $P$ lies outside the circle

$P$ lies outside if $PC > r$

$\Rightarrow \; x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0$

The minimum distance of $P$ from circle $= PA = CP - CA = |CP - r|$ $= |r - CP|$

and the maximum distance of $P$ from the circle $= PB = CP + CB$ $= |r + CP|$



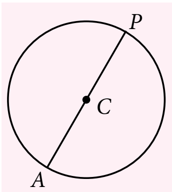

**Case III :** If $P$ lies on the circle

$P$ lies on the circle if $PC = r$

$\Rightarrow \; x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

In this case the minimum distance of $P$ from the circle $= 0 = |r - CP|$

and the maximum distance of $P$ from the circle $= PA = 2r = |r + CP|$




## TANGENT TO A CIRCLE AT A GIVEN POINT

- **Point Form**

  Equation of tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at $(x_1, y_1)$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

- **Parametric Form**

  Equation of tangent to circle $(x - \alpha)^2 + (y - \beta)^2 = a^2$ at the point $(\alpha + a\cos\theta, \beta + a\sin\theta)$ is $(x - \alpha)\cos\theta + (y - \beta)\sin\theta = a$.

- **Point of Intersection**

  The point of intersection of tangents at the points $P(\alpha)$ and $Q(\beta)$ on $x^2 + y^2 = a^2$ given by (where $\alpha$ and $\beta$ are parametric angle of the point $P$ and $Q$ respectively).

  $$\left(\frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}; \frac{a\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\right)$$

- **Slope Form**

  Equation of a line of slope $m$ *i.e.*, always tangent to the circle $(x - \alpha)^2 + (y - \beta)^2 = a^2$ is

  $$y - \beta = m(x - \alpha) \pm a\sqrt{1+m^2}$$

## TANGENTS FROM A POINT TO THE CIRCLE

From a given point two tangents can be drawn to a circle which are real, coincident or imaginary, according as the given point lies outside, on or inside the circle.

- The length of the tangent drawn from $(x_1, y_1)$ to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ is

  $$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

- Equation of the pair of tangents drawn from the point $(x_1, y_1)$ to the circle $S = 0$ is $T^2 = SS_1$.

## NORMAL

The normal of a circle at any point is a straight line which is perpendicular to the tangent at the point and always passes through the centre of the circle. The point of intersection of two normals or two diameters or two radii gives the centre of the circle.

## CHORD OF CONTACT

The chord joining the points of contact of the tangents drawn from an external point to any conic is known as the chord of contact w.r.t. that external point.

(i) Let $P(x_1, y_1)$ be a point outside the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$. Then the chord of contact of tangents drawn from $P$ to the circle is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ *i.e.*, $T = 0$.

(ii) If $R$ is the radius of a circle and tangents are drawn from a point $P$ to the circle then length of chord of contact is $\dfrac{2RL}{\sqrt{R^2+L^2}}$ where $L$ is the length of tangents.

(iii) The chord of contact of tangents drawn from a point $P$ to the circle $S = 0$ is always perpendicular to the line joining the centre of the circle to the point $P$.

## CHORD BISECTED AT A GIVEN POINT

(i) The equation of a chord of the circle $S = 0$ bisected at the point $P(x_1, y_1)$ is $S_1 = T$.

(ii) The length of the chord of the circle $S = 0$ bisected at the point $P(x_1, y_1)$ is $2\sqrt{-S_1}$.

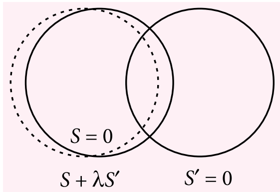
(iii) Equation of the chord joining two points $P(\alpha)$ and $Q(\beta)$ on the circle $x^2 + y^2 = a^2$ is $x\cos\left(\frac{\alpha+\beta}{2}\right) + y\sin\left(\frac{\alpha+\beta}{2}\right) = a\cos\left(\frac{\alpha-\beta}{2}\right)$ where $\alpha$ and $\beta$ are respectively parametric angle of the point $P$ and $Q$.

## COMMON CHORD OF TWO CIRCLES

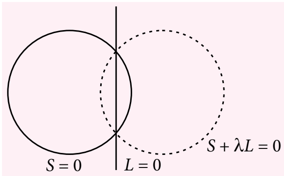
The chord joining the points of intersection of two given circles is called their common chord and its equation will be given by $S - S' = 0$ where $S$ and $S'$ are the equation of the two circles.

## FAMILY OF CIRCLES

(i) The equation of the family of circles passing through the point of intersection of two given circles $S = 0$ and $S' = 0$ is given as $S + \lambda S' = 0$ (where $\lambda$ is a parameter, $\lambda \in R - \{-1\}$)




(ii) The equation of the family of circles passing through the point of intersection of circle $S = 0$ and a line $L = 0$ is given as $S + \lambda L = 0$ (where $\lambda$ is a parameter and $\lambda \in R$)



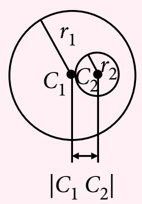
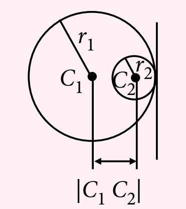
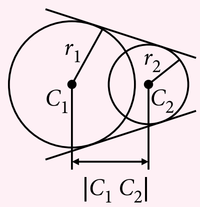
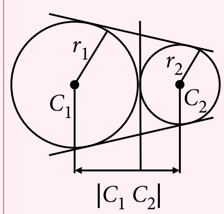
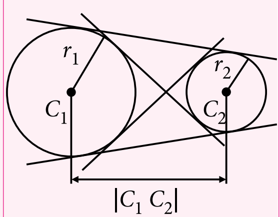

## RELATION BETWEEN TWO CIRCLES

Two circles may touch each other internally or externally, may lie inside or outside of each other, depending on these situations the relation between their radii and distance between the centres and number of common tangents varies. These situation are discussed as below :

**Common Tangent :** A line which touches both the circle is called common tangent. There are two types of common tangent

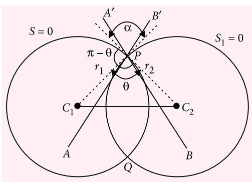
**(i) Direct Common Tangent :** When two circles touching the given line on same side of the line then the line is called the direct common tangent.

**(ii) Indirect Common Tangent :** When two circles touching the given line on different sides of the line then the line is called the indirect common tangent or transverse common tangent.

| | Smaller circle lies inside the bigger | Two circles touch internally | Two circles cut each other | Two circles touch externally | Two circles lies outside each other |
|---|---|---|---|---|---|
| Two or more circles in a plane |  $\lvert C_1 C_2\rvert$ |  $\lvert C_1 C_2\rvert$ |  $\lvert C_1 C_2\rvert$ |  $\lvert C_1 C_2\rvert$ |  $\lvert C_1 C_2\rvert$ |
| Relation betweeen centres and radii | $\lvert r_1 - r_2\rvert > \lvert C_1C_2\rvert$ | $\lvert r_1 - r_2\rvert = \lvert C_1C_2\rvert$ | $\lvert r_1 - r_2\rvert < \lvert C_1C_2\rvert < \lvert r_1 + r_2\rvert$ | $\lvert C_1C_2\rvert = \lvert r_1 + r_2\rvert$ | $\lvert C_1C_2\rvert > \lvert r_1 + r_2\rvert$ |
| Number of tangents | No Tangents | 1 DCT<br>0 TCT | 2 DCT<br>0 TCT | 2 DCT<br>1 TCT | 2 DCT<br>2 TCT |
| DCT $\Rightarrow$ Direct Common Tangents ; TCT $\Rightarrow$ Transverse Common Tangents | | | | | |

## ANGLE OF INTERSECTION OF TWO CIRCLES

Angle between two circles is defined as the angle between the tangents or between the normals of the two circles at the point of intersection.




If $\theta$ is the angle of intersection of circles having radii $r_1$ and $r_2$ and centres $C_1$ and $C_2$, such that the distance between their centres is $d$, then $\cos\theta = \dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2} = x$

## ORTHOGONAL CIRCLES

Two circles are said to be orthogonal if they intersect each other at right angles.

Given two circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x_2 + y_2 + 2g_2x + 2f_2y + c_2 = 0$. If the given circles are orthogonal then the required condition is $2g_1g_2 + 2f_1f_2 = c_1 + c_2$.

## RADICAL AXIS

- The radical axis of two circles is the locus of a point which moves such that the lengths of the tangents drawn from it to the two circles are equal.

- Equation of radical axis
Let $S_1 : x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S_2 : x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ be two equations of circles in general form. Then the equation of radical axis of two circles is given by $S_1 - S_2 = 0$

## RADICAL CENTRE

The radical axes of three circles whose centres are non-collinear, taken in pairs, meet in a point, which is called their radical centre. If the centres of the circles are collinear then their radical axis becomes parallel hence there is no point of intersection. *i.e.*, no radical centre.

## PROBLEMS

**1.** Equation of the circle having centre at (3, –1) and cutting the intercept of length 6 units on the line $2x - 5y + 18 = 0$ is
(a) $x^2 + y^2 - 6x + 2y - 18 = 0$
(b) $x^2 + y^2 - 6x + 2y - 38 = 0$
(c) $x^2 + y^2 - 6x + 2y - 28 = 0$
(d) None of these

**2.** Tangent to circle $x^2 + y^2 = 5$ at (1, –2) also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$. Co-ordinate of the corresponding point of contact is
(a) (3, 1) (b) (3, –1) (c) (–3, 1) (d) (–3, –1)

**3.** Tangents are drawn to $x^2 + y^2 = 1$ from any arbitrary point $P$ on the line $2x + y - 4 = 0$. The corresponding chord of contact passes through a fixed point whose coordinate is
(a) $\left(\frac{1}{4},\frac{1}{2}\right)$ (b) $\left(\frac{1}{2},1\right)$ (c) $\left(\frac{1}{2},\frac{1}{4}\right)$ (d) $\left(1,\frac{1}{2}\right)$

**4.** Equation of the circle drawn on the common chord of circle $x^2 + y^2 = 1$ and $x^2 + y^2 - 2x - 2y + 1 = 0$ as diameter is
(a) $x^2 + y^2 + x + y = 0$ (b) $x^2 + y^2 + x - y = 0$
(c) $x^2 + y^2 - x - y = 0$ (d) $x^2 + y^2 - x + y = 0$

**5.** Tangents $PA$ and $PB$ are drawn to $x^2 + y^2 = a^2$ from the point $P(x_1, y_1)$. Equation of the circumcircle of triangle $PAB$ is
(a) $x^2 + y^2 - xx_1 - yy_1 = 0$
(b) $x^2 + y^2 + xx_1 - yy_1 = 0$
(c) $x^2 + y^2 - xx_1 + yy_1 = 0$
(d) $x^2 + y^2 + xx_1 + yy_1 = 0$

**6.** Tangents $PA$ and $PB$ drawn to $x^2 + y^2 = 9$ from any arbitrary point '$P$' on the line $x + y = 25$. Locus of mid point of chord $AB$ is
(a) $25(x^2 + y^2) = 9(x + y)$
(b) $25(x^2 + y^2) = 3(x + 5)$
(c) $5(x^2 + y^2) = 3(x + y)$
(d) None of these

**7.** If the circle $x^2 + y^2 + 2ax + 2by = 0$ and $x^2 + y^2 + 2bx + 2cy = 0$ touch each other then
(a) $2b = a + c$ (b) $b = \frac{2ac}{a+c}$
(c) $b^2 = ac$ (d) $a + b + c = 0$

**8.** Any arbitrary tangent of $C_1 : x^2 + y^2 - a^2 = 0$ meets the circle $C_2 : x^2 + y^2 - 5a^2 = 0$ at the points $A$ and $B$. Locus of point of intersection of tangents drawn to $C_2$ at points $A$ and $B$ is
(a) $x^2 + y^2 = 20a^2$ (b) $x^2 + y^2 = 6a^2$
(c) $x^2 + y^2 = 25a^2$ (d) $x^2 + y^2 = 10a^2$

**9.** Orthocenter of the triangle $ABC$ where $A \equiv (a\cos\theta_1, a\sin\theta_1)$, $B \equiv (a\cos\theta_2, a\sin\theta_2)$ and $C \equiv (a\cos\theta_3, a\sin\theta_3)$ is
(a) $\left(\frac{2a}{3}\Sigma\cos\theta_1, \frac{2a}{3}\Sigma\sin\theta_1\right)$
(b) $\left(\frac{a}{2}\Sigma\cos\theta_1, \frac{a}{2}\Sigma\sin\theta_1\right)$
(c) $\left(a\Sigma\cos\theta_1, a\Sigma\sin\theta_1\right)$
(d) $\left(\frac{3a}{2}\Sigma\cos\theta_1, \frac{3a}{2}\Sigma\sin\theta_1\right)$

**10.** A point '$P$' moves in such a way that $\frac{PA}{PB} = \lambda$ where $\lambda \in (0, 1)$ is a constant and $A$, $B$ are fixed point such that $AB = a$. Locus of '$P$' is a circle whose diameter is equal to
(a) $\frac{a\lambda}{1-\lambda^2}$ (b) $\frac{a\lambda}{2(1-\lambda^2)}$
(c) $\frac{2a\lambda}{1-\lambda^2}$ (d) None of these

**11.** The circles $x^2 + y^2 - 2x - 4y = 0$ and $x^2 + y^2 - 8y - 4 = 0$ touch each other. The co-ordinates of the corresponding point of contact is
(a) (0, 2) (b) (0, 1) (c) (2, 0) (d) (1, 0)

**12.** Chord of contact of the tangent drawn to $x^2 + y^2 = a^2$ from any point on $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$ then
(a) $b^2 = ac$ (b) $a^2 = bc$ (c) $c^2 = ab$ (d) $abc = 1$

**13.** Equation of incircle of equilateral triangle $ABC$ where $B \equiv (2, 0)$, $C \equiv (4, 0)$ and $A$ lies in fourth quadrant is
(a) $x^2 + y^2 - 6x + \frac{2y}{\sqrt{3}} + 9 = 0$

(b) $x^2+y^2-6x-\frac{2y}{\sqrt{3}}+9=0$

(c) $x^2+y^2+6x+\frac{2y}{\sqrt{3}}+9=0$

(d) None of these

**14.** Locus of the centre of a circle that passes through $(a, b)$ and cuts the circle $x^2 + y^2 = a^2$ orthogonally is

(a) $2ax + 2by = a^2 + b^2$ (b) $2ax + by = a^2 + 2b^2$

(c) $ax + 2by = 2b^2 + a^2$ (d) $2ax + 2by = b^2 + 2a^2$

**15.** If the circle $x^2 + y^2 + 2a_1x + 2b_1y + c_1 = 0$ bisects the circumference of $x^2 + y^2 + 2a_2x + 2b_2y + c_2 = 0$ then

(a) $a_2(a_1 - a_2) + b_2(b_1 - b_2) + c_2 - c_1 = 0$

(b) $a_2(a_1 - a_2) + b_2(b_1 - b_2) + c_1 - c_2 = 0$

(c) $2a_2(a_1 - a_2) + 2b_2(b_1 - b_2) + c_2 - c_1 = 0$

(d) $2a_2(a_1 - a_2) + 2b_2(b_1 - b_2) + c_1 - c_2 = 0$

**16.** In triangle $ABC$ equation of side $BC$ is $x - y = 0$. Circumcenter and orthocenter of the triangle are (2, 3) and (5, 8) respectively. Equation of circumcircle of the triangle is

(a) $x^2 + y^2 - 4x + 6y - 27 = 0$

(b) $x^2 + y^2 - 4x - 6y - 27 = 0$

(c) $x^2 + y^2 + 4x + 6y - 27 = 0$

(d) $x^2 + y^2 + 4x - 6y - 27 = 0$

**17.** The line $y = mx + c$ cut the circle $x^2 + y^2 = a^2$ in the distinct points $A$ and $B$. Equation of the circle having minimum radius that can be drawn through the points $A$ and $B$ is

(a) $(1 + m^2)(x^2 + y^2 - a^2) + 2c(y - mx - c) = 0$

(b) $(1 + m^2)(x^2 + y^2 - a^2) + c(y - mx - c) = 0$

(c) $(1 + m^2)(x^2 + y^2 - a^2) - 2c(y - mx - c) = 0$

(d) $(1 + m^2)(x^2 + y^2 - a^2) - c(y - mx - c) = 0$

**18.** Equation of the smaller circle that touches the circle $x^2 + y^2 = 1$ and passes through the point (4, 3) is

(a) $5(x^2 + y^2) - 24x - 18y + 25 = 0$

(b) $x^2 + y^2 - 24x - 18y + 5 = 0$

(c) $5(x^2 + y^2) - 24x + 18y + 25 = 0$

(d) $5(x^2 + y^2) + 24x - 18y + 25 = 0$

**19.** If the angle between tangents drawn to $x^2 + y^2 + 2gx + 2fy + c = 0$ from (0, 0) is $\pi/2$, then

(a) $g^2 + f^2 = 3c$ (b) $g^2 + f^2 = 2c$

(c) $g^2 + f^2 = 5c$ (d) $g^2 + f^2 = 4c$

**20.** Co-ordinates of the midpoint of the segment cut by the circle $x^2 + y^2 - 6x + 2y - 54 = 0$ on the line $2x - 5y + 18 = 0$ is

(a) (4, 1) (b) (1, 3) (c) (1, 4) (d) (3, 1)

**21.** Combined equation of tangents drawn to $x^2 + y^2 = 1$ from the point (–1, –1) is

(a) $x + y - xy - 1 = 0$ (b) $x - y + xy + 1 = 0$

(c) $x - y - xy - 1 = 0$ (d) $x + y + xy + 1 = 0$

**22.** Equation of the circumcircle of equilateral triangle $ABC$ is $x^2 + y^2 + 2ax + 2by = 0$. If one vertex of the triangle coincides with origin then equation of incircle of triangle $ABC$ is

(a) $4x^2 + 4y^2 + 8ax + 8by + 2a^2 + 3b^2 = 0$

(b) $4x^2 + 4y^2 + 8ax + 8by + 3a^2 + 2b^2 = 0$

(c) $4x^2 + 4y^2 + 8ax + 8by + 2a^2 + 2b^2 = 0$

(d) $4x^2 + 4y^2 + 8ax + 8by + 3a^2 + 3b^2 = 0$

**23.** Two distinct chords of the circle $x^2 + y^2 - 2x - 4y = 0$ drawn from the point $P(a, b)$ gets bisected by the $y$-axis then

(a) $(b + 2)^2 > 4a$ (b) $(b - 2)^2 > 4a$

(c) $(b - 2)^2 > 5a$ (d) $(b + 2)^2 > 2a$

**24.** Locus of the centre of circle that cuts the circle $x^2 + y^2 + 2a_1x + 2b_1y + c_1 = 0$ and $x^2 + y^2 + 2a_2x + 2b_2y + c_2 = 0$ orthogonally is

(a) $2x(a_1 - a_2) + 2y(b_1 - b_2) + c_2 - c_1 = 0$

(b) $x(a_1 - a_2) + y(b_1 - b_2) + c_2 - c_1 = 0$

(c) $x(a_1 - a_2) + y(b_1 - b_2) + c_1 - c_2 = 0$

(d) $2x(a_1 - a_2) + 2y(b_1 - b_2) + c_1 - c_2 = 0$

**25.** Locus of midpoint of chords of circle $x^2 + y^2 = a^2$ that subtends angle $\pi/2$ at the point $(0, b)$ is

(a) $2x^2 + 2y^2 - 2bx + b^2 - a^2 = 0$

(b) $2x^2 + 2y^2 - 2by + b^2 - a^2 = 0$

(c) $2x^2 + 2y^2 - 2by + a^2 - b^2 = 0$

(d) $2x^2 + 2y^2 - 2by + a^2 + b^2 = 0$

**26.** If the circle $x^2 + y^2 + 2cx + b = 0$ and $x^2 + y^2 + 2cy + b = 0$ touch each other then

(a) $b > 0$ (b) $b < 0$

(c) $b = 0$ (d) None of these

**27.** Tangents $PA$ and $PB$ are drawn to $x^2 + y^2 = 4$ from the point $P(3, 0)$. Area of triangle $PAB$ is equal to

(a) $\frac{5}{9}\sqrt{5}$ sq. units (b) $\frac{1}{3}\sqrt{5}$ sq. units

(c) $\frac{10}{9}\sqrt{5}$ sq.units (d) $\frac{20}{3}\sqrt{5}$ sq. units

**28.** A light ray gets reflected from the $x = -2$. If the reflected ray touches the circle $x^2 + y^2 = 4$ and point of incident is $(-2, -4)$ then equation of incident ray is

(a) $4y + 3x + 22 = 0$ (b) $3y + 4x + 20 = 0$

(c) $4y + 2x + 20 = 0$ (d) $x + y + 6 = 0$

**29.** The circle $x^2 + y^2 + 2a_1x + c = 0$ lies completely inside the circle $x^2 + y^2 + 2a_2x + c = 0$, then

(a) $a_1a_2 > 0, c < 0$ (b) $a_1a_2 > 0, c > 0$
(c) $a_1a_2 < 0, c < 0$ (d) $a_1a_2 < 0, c > 0$

**30.** Radius of the circle that can be drawn to pass through the point (0, 1), (0, 6) and touching the $x$-axis is
(a) 5/2 (b) 13/2 (c) 7/2 (d) 9/2

**31.** A circle passes through the points $A(1, 0)$, $B(5, 0)$ and touches the $y$-axis at $C(0, h)$. If $\angle ACB$ is maximum, then $h =$
(a) $\sqrt{5}$ (b) $2\sqrt{5}$ (c) $\sqrt{10}$ (d) $2\sqrt{10}$

**32.** Consider four circles $(x \pm 1)^2 + (y \pm 1)^2 = 1$. Equation of smaller circle touching these four circles is
(a) $x^2+y^2=3-\sqrt{2}$ (b) $x^2+y^2=6-3\sqrt{2}$
(c) $x^2+y^2=5-2\sqrt{2}$ (d) $x^2+y^2=3-2\sqrt{2}$

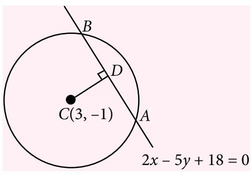
**33.** Radius of bigger circle touching the circle $x^2 + y^2 - 4x - 4y + 4 = 0$ and both the co-ordinate axis is
(a) $(3+2\sqrt{2})$ (b) $2(3+2\sqrt{2})$
(c) $(6+2\sqrt{2})$ (d) $2(6+2\sqrt{2})$

**34.** $f(x, y) = x^2 + y^2 + 2ax + 2by + c = 0$ represents a circle. If $f(x, 0) = 0$ has equal roots, each being 2 and $f(0, y) = 0$ has 2 and 3 as it's roots, then centre of circle is
(a) $\left(2, \frac{5}{2}\right)$ (b) $\left(-2, -\frac{5}{2}\right)$
(c) Data is not sufficient (d) None of these

**35.** Circles are drawn having the sides of triangle $ABC$ as their diameters. A radical centre of these circles is the
(a) circumcenter of triangle $ABC$
(b) incenter of triangle $ABC$
(c) orthocenter of triangle $ABC$
(d) centroid of triangle $ABC$

## SOLUTIONS

**1.** **(c) :** Here, $CD = \frac{|6+5+18|}{\sqrt{4+25}} = \sqrt{29}$
$AD = 3$
$\Rightarrow$ $CA^2 = 9 + 29 = 38$
Thus equation of circle is,
$(x - 3)^2 + (y + 1)^2 = 38$
*i.e.*, $x^2 + y^2 - 6x + 2y - 28 = 0$

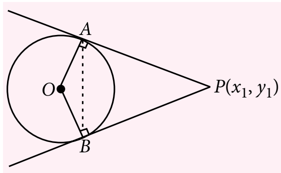



**2.** **(b) :** Equation of tangent to $x^2 + y^2 = 5$ at $(1, -2)$ is $x - 2y = 5$.
Putting $x = 2y + 5$ in second circle, we get
$(2y + 5)^2 + y^2 - 8(2y + 5) + 6y + 20 = 0$
$\Rightarrow$ $5y^2 + 10y + 5 = 0$
$\Rightarrow$ $y = -1$ $\Rightarrow$ $x = -2 + 5 = 3$
Thus point of contact is $(3, -1)$

**3.** **(c) :** Let $P \equiv (a, 4 - 2a)$
Equation of chord of contact is
$x \cdot a + y \cdot (4 - 2a) = 1 \Rightarrow (4y - 1) + a(x - 2y) = 0$
It will always pass through a fixed point whose coordinates are $y = \frac{1}{4}$ and $x = 2y = \frac{1}{2}$

**4.** **(c) :** Equation of common chord is
$2 - 2x - 2y = 0 \Rightarrow x + y = 1$
It meet $x^2 + y^2 = 1$ at (1, 0), (0, 1).
Thus equation of circle is $(x - 1)x + y(y - 1) = 0$
*i.e.*, $x^2 + y^2 - x - y = 0$

**5.** **(a) :** Clearly, the points $O$, $A$, $P$ and $B$ are concyclic and midpoint of $OP$ is the centre of this circle.
Thus, equation of circumcircle of triangle $PAB$ is




$x(x - x_1) + y(y - y_1) = 0$ *i.e.*, $x^2 + y^2 - xx_1 - yy_1 = 0$

**6.** **(a) :** Let $P \equiv (a, 25 - a)$
Equation of chord $AB$ is $T = 0$
*i.e.*, $xa + y(25 - a) = 9$
If midpoint of chord $AB$ is $C(h, k)$, then equation of chord $AB$ is $T = S_1$. *i.e.*, $xh + yk = h^2 + k^2$
Comparing the coefficients, we get

$$\frac{a}{h} = \frac{25-a}{k} = \frac{9}{h^2+k^2} = \frac{a+25-a}{h+k} = \frac{25}{h+k}$$

Thus locus of '$C$' is $25(x^2 + y^2) = 9(x + y)$

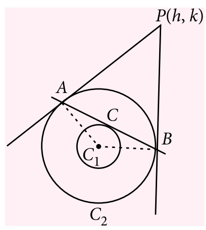
**7.** **(c) :** Both circles passing through origin. If they touch each other then tangents drawn to respective circles at origin must be identical lines.
*i.e.*, $ax + by = 0$ and $bx + cy = 0$ should represent same line. $\Rightarrow$ $\frac{a}{b} = \frac{b}{c}$ $\Rightarrow$ $b^2 = ac$

**8.** **(c) :** Let $AB$ be a tangent to $C_1$, drawn at the point $C(a \cos\theta, a \sin\theta)$ and tangents drawn to $C_2$ at $A$ and $B$, intersect at $P(h, k)$.
Then equation of $AB$ is, $x \cos\theta + y \sin\theta = a$
Also the equation of line $AB$ is, $xh + yk = 5a^2$
Comparing the coefficients, we get

$$\frac{\cos\theta}{h} = \frac{\sin\theta}{k} = \frac{a}{5a^2}$$

$$\Rightarrow \cos\theta = \frac{h}{5a}, \ \sin\theta = \frac{k}{5a}$$

$\Rightarrow$ $h^2 + k^2 = 25a^2$




Thus required locus is $x^2 + y^2 = 25a^2$

**9.** **(c) :** $A$, $B$ and $C$ lies on the circle $x^2 + y^2 = a^2$.
That means circumcenter of triangle $ABC$ is

$O \equiv (0, 0)$ and it's centroid is $G \equiv \left(\frac{a}{3}\Sigma\cos\theta_1, \frac{a}{3}\Sigma\sin\theta_1\right)$

If orthocenter of triangle be $H(x_P, y_P)$, then

$OG : GH = 1 : 2$

$\Rightarrow \quad x_P = a\,\Sigma\cos\theta_1$ and $y_P = a\,\Sigma\sin\theta_1$.

**10. (c) :** Let $A \equiv (0, 0)$, $B \equiv (a, 0)$, $P \equiv (x, y)$

We have $PA^2 = \lambda^2 PB^2$

$\Rightarrow \quad x^2 + y^2 = \lambda^2((x - a)^2 + y^2)$

$\Rightarrow \quad x^2(1 - \lambda^2) + y^2(1 - \lambda^2) + 2a\lambda^2 x - a^2\lambda^2 = 0$

It is a circle, whose radius

$$= \sqrt{\frac{a^2\lambda^4}{(1-\lambda^2)^2} + \frac{a^2\lambda^2}{(1-\lambda^2)}} = \frac{a\lambda}{(1-\lambda^2)}$$

Thus it's diameter $= \dfrac{2a\lambda}{1-\lambda^2}$

**11. (c) :** Equation of common tangent is

$S_1 - S_2 = 0$ *i.e.*, $4y - 2x + 4 = 0$ *i.e.*, $x = 2y + 2$

Solving it with $x^2 + y^2 - 2x - 4y = 0$, we get

$(2y + 2)^2 + y^2 - 2(2y + 2) - 4y = 0$

$\Rightarrow \quad 5y^2 = 0 \quad \Rightarrow \quad y = 0 \quad \therefore \quad x = 2$

Thus point of contact is $(2, 0)$

**12. (b) :** Let $P(x_1, y_1)$ be any point on $x^2 + y^2 = b^2$

*i.e.*, $x_1^2 + y_1^2 = b^2$

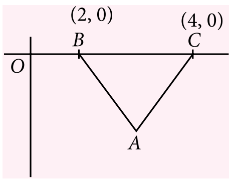
Equation of corresponding chord of contact is

$xx_1 + yy_1 - a^2 = 0$

Since, it touches $x^2 + y^2 = c^2$,

$$\therefore \quad \frac{|-a^2|}{\sqrt{x_1^2 + y_1^2}} = |c| \Rightarrow a^2 = |bc|$$

**13. (a) :** Clearly, $A \equiv (3, -\sqrt{3})$




Centroid of $\Delta ABC$ is $\left(3, -\frac{1}{\sqrt{3}}\right)$

Thus equation of incircle is,

$$(x-3)^2 + \left(y + \frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3} \Rightarrow x^2 + y^2 - 6x + \frac{2y}{\sqrt{3}} + 9 = 0$$

**14. (d) :** Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Since, it passes through $(a, b)$, thus

$a^2 + b^2 + 2ag + 2fb + c = 0$

It also cuts $x^2 + y^2 = a^2$ orthogonally, thus

$2g\cdot 0 + 2f\cdot 0 = c - a^2 \quad \Rightarrow \quad c = a^2$

$\Rightarrow \quad 2ag + 2fb + b^2 + 2a^2 = 0$

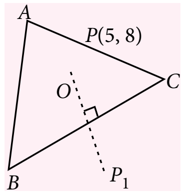
Thus locus of centre is $2ax + 2by = b^2 + 2a^2$

**15. (c) :** Clearly the centre of second circle *i.e.*, $(-a_2, -b_2)$ should lie on the common chord of circles

$2(a_1 - a_2)x + 2(b_1 - b_2)y + c_1 - c_2 = 0$

$\therefore \quad 2a_2(a_1 - a_2) + 2b_2(b_1 - b_2) + c_2 - c_1 = 0$

**16. (b) :** Reflection of '$P$' in $BC$ will lie on circumcircle.




Clearly $P_1 \equiv (8, 5)$

Thus equation of circumcircle is

$(x - 2)^2 + (y - 3)^2 = (8 - 2)^2 + (5 - 3)^2$

*i.e.*, $x^2 + y^2 - 4x - 6y - 27 = 0$

**17. (c) :** Equation of required circle will be in the form of $x^2 + y^2 - a^2 + \lambda(y - mx - c) = 0$

If it's radius is minimum then it's centre *i.e.*,

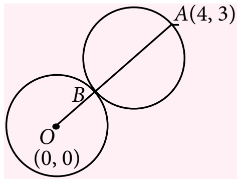
$\left(\frac{m\lambda}{2}, -\frac{\lambda}{2}\right)$ must lie on the line $y = mx + c$

$$\Rightarrow \quad -\frac{\lambda}{2} = \frac{m^2\lambda}{2} + c \quad \Rightarrow \quad \lambda = \frac{-2c}{(1+m^2)}$$

Thus required circle is

$(1 + m^2)(x^2 + y^2 - a^2) - 2c(y - mx - c) = 0$

**18. (a) :** For smallest circle $OA$ will become common normal.




$OA = 5 \quad \Rightarrow \quad AB = 4$

Equation of line $OA$ is $y = \frac{3}{4}x$

Put $y = \frac{3}{4}x$ in $x^2 + y^2 = 1$.

We get $x^2 + \frac{9x^2}{16} = 1 \Rightarrow \quad x = \pm\frac{4}{5} \quad \Rightarrow \quad B \equiv \left(\frac{4}{5}, \frac{3}{5}\right)$

Thus required circle is

$$\left(x - \frac{4}{5}\right)(x-4) + (y-3)\left(y - \frac{3}{5}\right) = 0$$

$\Rightarrow \quad 5x^2 + 5y^2 - 24x - 18y + 25 = 0$

**19. (b) :** Clearly, $(0, 0)$ lies on director circle of the given circle.

Now, equation of director circle is

$(x + g)^2 + (y + f)^2 = 2(g^2 + f^2 - c)$

If $(0, 0)$ lies on it, then $g^2 + f^2 = 2(g^2 + f^2 - c)$

$\Rightarrow \quad g^2 + f^2 = 2c$

**20. (c) :** Centre of circle is $(3, -1)$.

Equation of any line passing through $(3, -1)$ and perpendicular to $2x - 5y + 18 = 0$ is,

$(y+1) = -\frac{5}{2}(x-3)$ *i.e.*, $2y + 5x - 13 = 0$

Solving with $2x - 5y + 18 = 0$

Mid point is $(1, 4)$.

**21. (d) :** Tangents are clearly, $x = -1$, $y = -1$

Their combined equation is $(x + 1)(y + 1) = 0$

*i.e.*, $x + y + xy + 1 = 0$

**22. (d) :** Centre is $(-a, -b)$ and radius is $\sqrt{a^2 + b^2}$.

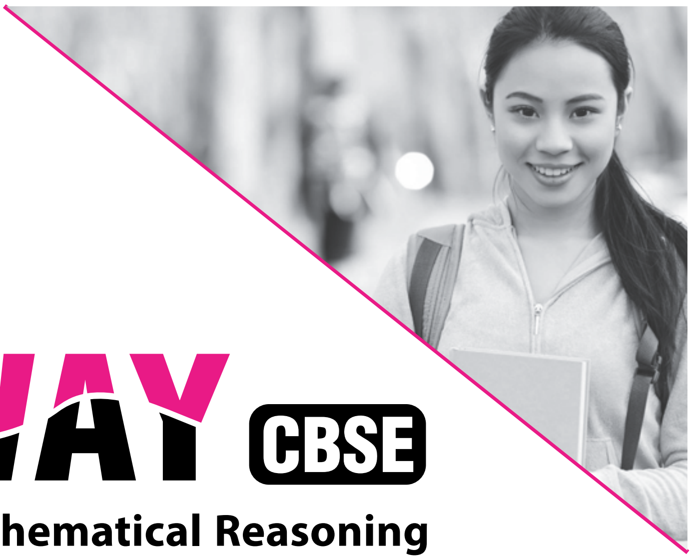
Thus equation of incircle is $(x+a)^2 + (y+b)^2 = \dfrac{a^2+b^2}{4}$

$\Rightarrow \quad 4x^2 + 4y^2 + 8ax + 8by + 3a^2 + 3b^2 = 0$

*Contd. on Page No. 81*

CLASS XI | Series 8

# ACE YOUR WAY CBSE



## Limits and Derivatives | Mathematical Reasoning

### HIGHLIGHTS

### LIMITS AND DERIVATIVES

| | Left Hand Limit | Right Hand Limit |
|---|---|---|
| **Definition** | A real number $L$ is the left hand limit of a function $f(x)$ if $x$ approaches to a point $a$ to the left of $a$ then $f(x)$ approaches to $L$. | A real number $L$ is the right hand limit of a function $f(x)$ if $x$ approaches to a point $a$ to the right of $a$ then $f(x)$ approaches to $L$. |
| **Notation** | $\lim\limits_{x \to a^-} f(x) = L$ | $\lim\limits_{x \to a^+} f(x) = L$ |

### LIMIT OF A FUNCTION

Limit of a function at a point is the common value of the left and right hand limits, if they coincide. The limit of a function $f(x)$ at $x = a$ is denoted as $\lim\limits_{x \to a} f(x)$.

$\therefore$ Limit of a function exists iff both left hand limit and right hand limit exists and equal.

*i.e.* $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$

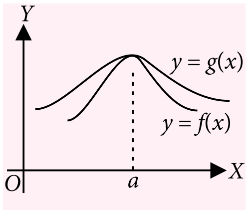
### Algebra of Limits

If $f(x)$ and $g(x)$ be two functions such that $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exists. Then,

| | Definition | Representation |
|---|---|---|
| **Sum** | Limit of sum of two functions is the sum of the limits of the functions. | $\lim\limits_{x \to a}[f(x)+g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$ |
| **Difference** | Limit of difference of two functions is the difference of the limits of the functions. | $\lim\limits_{x \to a}[f(x)-g(x)] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$ |
| **Product** | Limit of product of two functions is the product of limits of the functions. | $\lim\limits_{x \to a}[f(x)\cdot g(x)] = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$ |
| **Division** | Limit of quotient of two functions is the quotient of limits of the functions, when limit of denominator is non-zero. | $\lim\limits_{x \to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$; $\lim\limits_{x \to a} g(x) \neq 0$ |

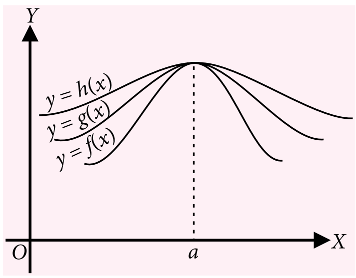
- **Some Important Theorems**

1. If $f$ and $g$ be two real valued functions such that $f(x) \le g(x) \forall x$ lies in the common domain of $f$ and $g$, then $\lim\limits_{x \to a} f(x) \le \lim\limits_{x \to a} g(x)$,




when both $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist at point '$a$'.

2. **Sandwich Theorem :** If $f(x)$, $g(x)$ and $h(x)$ are real functions of $x$ such that $f(x) \le g(x) \le h(x)$ for all $x$ lies in the common domain of $f$, $g$ and $h$ and for some real number $a$, $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x) = l$ (say), then $\lim\limits_{x \to a} g(x) = l$.




### Some Standard Limits

- $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$, for any positive integer $n$.
- $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$
- $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x} = 0$
- $\lim\limits_{x \to a} \dfrac{\sin(x-a)}{x-a} = 1$
- $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1$
- $\lim\limits_{x \to a} \dfrac{\tan(x-a)}{x-a} = 1$
- $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$
- $\lim\limits_{x \to 0} \dfrac{\log_e(1+x)}{x} = 1$

## DERIVATIVES

Let $y = f(x)$ be a real valued function and $a$ is a point in its domain of definition, then derivative of the function at $a$ is given by

$f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ Provided that this limit exist.

## FIRST PRINCIPLE OF DERIVATIVE

Suppose $f$ is a real valued function, the function defined by $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, wherever the limit exists is defined to be the derivative of $f$ at $x$ and is denoted by $f'(x)$ or $\dfrac{dy}{dx}$. This definition of derivative is also called the first principle of derivative.

Thus, $f'(x)$ or $\dfrac{dy}{dx} = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

## ALGEBRA OF DERIVATIVE OF FUNCTIONS

Lef $f(x)$ and $g(x)$ be two functions such that their derivatives are defined. Then,

| | |
|---|---|
| **Sum** | $\dfrac{d}{dx}[f(x)+g(x)] = \dfrac{d}{dx}f(x) + \dfrac{d}{dx}g(x)$ |
| **Difference** | $\dfrac{d}{dx}[f(x)-g(x)] = \dfrac{d}{dx}f(x) - \dfrac{d}{dx}g(x)$ |
| **Product** | $\dfrac{d}{dx}[f(x)\cdot g(x)] =$ $f(x)\dfrac{d}{dx}g(x) + g(x)\dfrac{d}{dx}f(x)$ |
| **Quotient** | $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] =$ $\dfrac{g(x)\dfrac{d}{dx}f(x) - f(x)\dfrac{d}{dx}g(x)}{[g(x)]^2}, g(x) \ne 0$ |

### Some Standard Derivatives

- $\dfrac{d}{dx}(x^n) = nx^{n-1}$
- $\dfrac{d}{dx}(\sin x) = \cos x$
- $\dfrac{d}{dx}(\cos x) = -\sin x$

## STATEMENT

A statement is a sentence which is either true or false but not both.

## NEGATION OF A STATEMENT

A denial of a statement is called the negation of the statement.

## SIMPLE STATEMENT

A statement which has no other statement as its component.

## COMPOUND STATEMENT

A statement that can be formed by combining two or more simple statements by logical connectives (and, or etc.).

**Rule 1 :** The compound statement with "AND" is

- true, if all its component statements are true.
- false, if any of its component statements are false.

**Rule 2 :** The compound statement with an "OR" is

- true, when one component statement is true or both the component statements are true.
- false, when both component statements are false.

## QUANTIFIERS

Many mathematical statements contain phrases 'there exists' and 'for all' or 'for every'. These phrases are called quantifiers.

## IMPLICATIONS

In Mathematics, we come across many statements of the form "if then", "only if" and "if and only if", such statements are called implications.

- **If then implication :** For two statements $p$ and $q$, a sentence "if $p$ then $q$" can be written in the following ways :
  (i) $p$ implies $q$ (denoted by $p \Rightarrow q$)
  (ii) $p$ is sufficient condition for $q$
  (iii) $q$ is necessary condition for $p$
  (iv) $p$ only if $q$
  (v) $\sim q$ implies $\sim p$
- **If and only if Implication :** If $p$ and $q$ are two statements, then the compound statement $p \Rightarrow q$ and $q \Rightarrow p$ is called if and only if implication and is denoted by $p \Leftrightarrow q$.

## CONVERSE STATEMENT

If $p$ and $q$ are two statements, then the converse of the implication "if $p$ then $q$" is "if $q$ then $p$".

## CONTRAPOSITIVE STATEMENT

If $p$ and $q$ are two statements, then the contrapositive of the implication "if $p$ then $q$" is "if not $q$ then not $p$" *i.e.*, "if $\sim q$ then $\sim p$".

## INVERSE

If $p$ and $q$ are two statements, then the inverse of "if $p$ then $q$" is "if $\sim p$ then $\sim q$".

## PROBLEMS

### Very Short Answer Type

1. Evaluate : $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}+\sqrt{1-x}}{1+x}$
2. Write the negation of the following statement :
   All mathematicians are man.
3. Differentiate $(x^2 - 3x + 2)(x + 2)$ with respect to $x$.
4. Evaluate : $\lim\limits_{x\to 0} \dfrac{\sin 5x}{2x}$
5. Differentiate $\sin(x + a)$ with respect to $x$.

### Short Answer Type

6. Evaluate : $\lim\limits_{x\to 2}\left(\dfrac{1}{x-2}-\dfrac{4}{x^3-2x^2}\right)$
7. Check whether the following statement is true or not :
   If $x, y \in Z$ are such that $x$ and $y$ are odd, then $xy$ is odd.
8. Find the derivative of $\left(x^2+\dfrac{1}{x^2}\right)^3$ w.r.t. $x$.
9. Find $\lim\limits_{x\to 5} f(x)$, where $f(x) = |x| - 5$.
10. Write the component statements of the following compound statements and check whether the compound statement is true or false :
    (i) 125 is a multiple of 7 or 8.
    (ii) Mumbai is the capital of Gujarat or Maharashtra.

### Long Answer Type - I

11. Find the derivative of the following :
    (i) $\dfrac{1+\dfrac{1}{x}}{1-\dfrac{1}{x}}$ (ii) $\operatorname{cosec} x \cot x$
12. For what integers $m$ and $n$ does both $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 1} f(x)$ exist.

$$f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$$

**13.** Find the derivative of $f(x) = 2x^2 + 3x - 5$ using first principle at $x = 0$ and $x = -1$. Also, show that $f'(0) + 3f'(-1) = 0$.

**14.** Evaluate :

**(i)** $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{\sin^3 x}$ **(ii)** $\lim\limits_{x\to \frac{\pi}{2}}\dfrac{\cot x-\cos x}{(\pi-2x)^3}$

**15.** Find the derivative of $\sqrt{2x+3}$ from first principle.

## Long Answer Type - II

**16.** Evaluate : $\lim\limits_{y\to 0}\dfrac{(x+y)\sec(x+y)-x\sec x}{y}$

**17.** Evaluate :

**(i)** $\lim\limits_{x\to 2}\dfrac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}$

**(ii)** $\lim\limits_{x\to 4}\dfrac{3-\sqrt{5+x}}{1-\sqrt{5-x}}$

**18.** Differentiate $\sin(x^2+1)$ with respect to $x$ using first principle.

**19.** If $y = 1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+....+\dfrac{x^n}{n!}$, show that $\dfrac{dy}{dx}-y+\dfrac{x^n}{n!}=0.$

**20.** Evaluate : $\lim\limits_{x\to \frac{\pi}{6}}\dfrac{2-\sqrt{3}\cos x-\sin x}{(6x-\pi)^2}$

## SOLUTIONS

**1.** Here, $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}+\sqrt{1-x}}{1+x}$

$$=\frac{\sqrt{1+0}+\sqrt{1-0}}{1+0}=\frac{1+1}{1}=2$$

**2.** The negation of the given statement is :
Some mathematicians are not man.

**3.** Clearly, $\dfrac{d}{dx}\{(x^2-3x+2)(x+2)\}$

$$=\frac{d}{dx}(x^3-x^2-4x+4)$$

$$=\frac{d}{dx}(x^3)-\frac{d}{dx}(x^2)-\frac{d}{dx}(4x)+\frac{d}{dx}(4)$$

$= 3x^2 - 2x - 4$

**4.** Here, $\lim\limits_{x\to 0}\dfrac{\sin 5x}{2x}=\lim\limits_{x\to 0}\left(\dfrac{\sin 5x}{5x}\times\dfrac{5x}{2x}\right)$

$$=\frac{5}{2}\lim_{x\to 0}\frac{\sin 5x}{5x}=\frac{5}{2}\times 1=\frac{5}{2}$$

**5.** Clearly, $\dfrac{d}{dx}\{\sin(x+a)\}$

$$=\frac{d}{dx}\{\sin x\cos a+\cos x\sin a\}$$

$$=\frac{d}{dx}(\sin x\cos a)+\frac{d}{dx}(\cos x\sin a)$$

$$=\cos a\frac{d}{dx}(\sin x)+\sin a\frac{d}{dx}(\cos x)$$

$= \cos x \cos a - \sin x \sin a = \cos(x + a)$

**6.** $\lim\limits_{x\to 2}\left[\dfrac{1}{x-2}-\dfrac{4}{x^3-2x^2}\right]=\lim\limits_{x\to 2}\left[\dfrac{1}{x-2}-\dfrac{4}{x^2(x-2)}\right]$

$$=\lim_{x\to 2}\left[\frac{x^2-4}{x^2(x-2)}\right]=\lim_{x\to 2}\left[\frac{(x+2)(x-2)}{x^2(x-2)}\right]$$

$$=\frac{2+2}{2^2}=\frac{4}{4}=1$$

**7.** Let $p : x, y \in Z$ such that $x$ and $y$ are odd.
$q : xy$ is odd.
We have to check the statement 'if $p$ then $q$' is true or not. Let us assume that $p$ is true, then we will show that $q$ is true.
Let $x = 2m + 1$ and $y = 2n + 1$, for some integer $m$ and $n$. Thus,
$xy = (2m + 1)(2n + 1) = 2(2mn + m + n) + 1$
This shows that $xy$ is odd *i.e.*, $q$ is true. Therefore, the given statement is true.

**8.** Here, $\dfrac{d}{dx}\left(x^2+\dfrac{1}{x^2}\right)^3=\dfrac{d}{dx}\left[x^6+\dfrac{1}{x^6}+3x^2+\dfrac{3}{x^2}\right]$

$$=\frac{d}{dx}(x^6)+\frac{d}{dx}\left(x^{-6}\right)+3\frac{d}{dx}(x^2)+3\frac{d}{dx}(x^{-2})$$

$= 6x^5 - 6x^{-7} + 3 \times 2x + 3 \times (-2)x^{-3}$

$$=6x^5-\frac{6}{x^7}+6x-\frac{6}{x^3}$$

**9.** When $x > 5$, put $x = 5 + h$, where $h$ is small.
$\Rightarrow |x| = |5 + h| = 5 + h$
$\therefore \lim\limits_{x\to 5^+} f(x)=\lim\limits_{h\to 0}[(5+h)-5]=\lim\limits_{h\to 0} h=0$
When $x < 5$, put $x = 5 - h$, where $h$ is small.
$\Rightarrow |x| = |5 - h| = 5 - h$
$\therefore \lim\limits_{x\to 5^-} f(x)=\lim\limits_{h\to 0}(5-h-5)=\lim\limits_{h\to 0}(-h)=0$

$\therefore \lim_{x\to 5^-} f(x) = \lim_{h\to 5^+} f(x) = 0$

$\therefore \lim_{x\to 5} f(x) = 0$

**10.** **(i)** The component statements of the given statement are :

$p$ : 125 is a multiple of 7.

$q$ : 125 is a multiple of 8.

We observe that both $p$ and $q$ are false statements. Therefore, the compound statement is also false.

**(ii)** The component statements of the given statement are :

$p$ : Mumbai is the Capital of Gujarat.

$q$ : Mumbai is the Capital of Maharashtra.

We find that $p$ is false and $q$ is true. Therefore, the compound statement is true.

**11.** **(i)** Let $f(x) = \dfrac{1+\frac{1}{x}}{1-\frac{1}{x}} = \dfrac{x+1}{x-1}$

$$\therefore f'(x) = \frac{d}{dx}\left[\frac{x+1}{x-1}\right]$$

$$= \frac{(x-1)\frac{d}{dx}(x+1)-(x+1)\frac{d}{dx}(x-1)}{(x-1)^2}$$

$$= \frac{(x-1)\times 1-(x+1)\times 1}{(x-1)^2} = \frac{x-1-x-1}{(x-1)^2}$$

$$= \frac{-2}{(x-1)^2},\ x \neq 0,\ 1$$

**(ii)** Let $f(x) = \text{cosec}\, x \cot x = \dfrac{1}{\sin x}\times\dfrac{\cos x}{\sin x} = \dfrac{\cos x}{\sin^2 x}$

Then, $f'(x) = \dfrac{d}{dx}\left[\dfrac{\cos x}{\sin^2 x}\right]$

$$= \frac{\sin^2 x\frac{d}{dx}[\cos x]-\cos x\frac{d}{dx}[\sin^2 x]}{(\sin^2 x)^2}$$

$$= \frac{\sin^2 x(-\sin x)-\cos x\frac{d}{dx}(\sin x)(\sin x)}{\sin^4 x}$$

$$= \frac{-\sin^3 x-\cos x\left[\sin x\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}(\sin x)\right]}{\sin^4 x}$$

$$= \frac{-\sin^3 x-\cos x(\sin x\cos x+\sin x\cos x)}{\sin^4 x}$$

$$= \frac{-\sin^3 x-2\sin x\cos^2 x}{\sin^4 x}$$

$$= \frac{-\sin x\,[\sin^2 x+2\cos^2 x]}{\sin^4 x} = \frac{-\sin^2 x-2\cos^2 x}{\sin^3 x}$$

$= -\text{cosec}\, x - 2\cot^2 x\,\text{cosec}\, x$

$= -\text{cosec}\, x\,[1 + \cot^2 x + \cot^2 x]$

$= -\text{cosec}\, x\,[\text{cosec}^2 x + \cot^2 x]$

$= -\text{cosec}^3 x - \cot^2 x\,\text{cosec}\, x.$

**12.** **(a)** When $x < 0$, $f(x) = mx^2 + n$

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(mx^2+n) = n$

When $x > 0$, $f(x) = nx + m$

$\therefore \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(nx+m) = m$

For $\lim_{x\to 0} f(x)$ exist,

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

$\Rightarrow n = m$ ...(i)

**(b)** When $x < 1$, $f(x) = nx + m$

$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(nx+m) = n+m$

When $x > 1$, $f(x) = nx^3 + m$

$\therefore \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(nx^3+m) = n+m$

For $\lim_{x\to 1} f(x)$ exist,

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = m + n$ ...(ii)

Thus from (i) and (ii), both the limits exist for all equal integral values of $m$ and $n$.

**13.** Here, $f'(0) = \lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$

$$= \lim_{h\to 0}\frac{(2h^2+3h-5)-(-5)}{h} = \lim_{h\to 0}\frac{2h^2+3h}{h}$$

$= \lim_{h\to 0}(2h + 3) = 3$

Also, $f'(-1) = \lim_{h\to 0}\dfrac{f(-1+h)-f(-1)}{h}$

$$= \lim_{h\to 0}\frac{[2(-1+h)^2+3(-1+h)-5]-[2(-1)^2+3(-1)-5]}{h}$$

$$= \lim_{h\to 0}\frac{[2h^2+2-4h-3+3h-5]-[2-3-5]}{h}$$

$$= \lim_{h\to 0}\frac{2h^2-h-6+6}{h} = \lim_{h\to 0}\frac{2h^2-h}{h}$$

$= \lim_{h\to 0}(2h - 1) = -1$

Now, $f'(0) + 3f'(-1) = 3 + 3\times(-1) = 3 - 3 = 0.$

**14. (i)** We have, $\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}-\sin x}{\sin^3 x}$

$$=\lim_{x\to 0}\frac{\sin x(1-\cos x)}{\cos x\ \sin^3 x}=\lim_{x\to 0}\frac{1-\cos x}{\cos x\ \sin^2 x}$$

$$=\lim_{x\to 0}\frac{1-\cos x}{(\cos x)(1-\cos^2 x)}$$

$$=\lim_{x\to 0}\frac{1}{\cos x(1+\cos x)}=\frac{1}{1\cdot(1+1)}=\frac{1}{2}$$

**(ii)** We have, $\lim_{x\to\pi/2}\frac{\cot x-\cos x}{(\pi-2x)^3}$

$$=\lim_{h\to 0}\frac{\cot\left(\frac{\pi}{2}+h\right)-\cos\left(\frac{\pi}{2}+h\right)}{\left\{\pi-2\left(\frac{\pi}{2}+h\right)\right\}^3}$$

$$=\lim_{h\to 0}\frac{-\tan h+\sin h}{-8h^3}=\lim_{h\to 0}\frac{-\sin h(1-\cos h)}{\cos h(-8h^3)}$$

$$=\frac{1}{8}\lim_{h\to 0}\frac{\tan h}{h}\times\frac{1-\cos h}{h^2}$$

$$=\frac{1}{8}\lim_{h\to 0}\frac{\tan h}{h}\times\lim_{h\to 0}\frac{2\sin^2\frac{h}{2}}{h^2}$$

$$=\frac{1}{8}\times\frac{2}{4}\lim_{h\to 0}\frac{\sin^2\frac{h}{2}}{\left(\frac{h}{2}\right)^2}=\frac{1}{16}$$

**15.** Let, $f(x)=\sqrt{2x+3}$

Then $f(x+h)=\sqrt{2(x+h)+3}=\sqrt{2x+2h+3}$

We know that $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\Rightarrow f'(x)=\lim_{h\to 0}\frac{\sqrt{2x+2h+3}-\sqrt{2x+3}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{2x+2h+3}-\sqrt{2x+3}}{h}\times\frac{\sqrt{2x+2h+3}+\sqrt{2x+3}}{\sqrt{2x+2h+3}+\sqrt{2x+3}}$$

$$=\lim_{h\to 0}\frac{2x+2h+3-2x-3}{h(\sqrt{2x+2h+3}+\sqrt{2x+3})}$$

$$=\lim_{h\to 0}\frac{2h}{h(\sqrt{2x+2h+3}+\sqrt{2x+3})}$$

$$=\frac{2}{2\sqrt{2x+3}}=\frac{1}{\sqrt{2x+3}}$$

**16.** $\lim_{y\to 0}\frac{(x+y)\sec(x+y)-x\sec x}{y}$

$$=\lim_{y\to 0}\frac{x\sec(x+y)-x\sec x+y\sec(x+y)}{y}$$

$$=\lim_{y\to 0}\frac{x[\sec(x+y)-\sec x]}{y}+\sec(x+y)$$

$$=\lim_{y\to 0}\frac{x\left(\frac{1}{\cos(x+y)}-\frac{1}{\cos x}\right)}{y}+\lim_{y\to 0}\sec(x+y)$$

$$=\lim_{y\to 0}x\times\frac{\cos x-\cos(x+y)}{y\cos(x+y)\cos x}+\sec x$$

$$=\lim_{y\to 0}x\times\frac{2\sin\left(x+\frac{y}{2}\right)\sin\frac{y}{2}}{y\cos(x+y)\cos x}+\sec x$$

$$=x\times\frac{\lim_{y\to 0}\sin\left(x+\frac{y}{2}\right)}{\cos x\lim_{y\to 0}\cos(x+y)}\cdot\lim_{y/2\to 0}\frac{\sin(y/2)}{y/2}+\sec x$$

$$=\frac{x\sin x}{\cos x\cos x}\cdot 1+\sec x=x\tan x\sec x+\sec x$$

**17. (i)** Here, $\lim_{x\to 2}\frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}$

Rationalising the denominator, we have

$$=\lim_{x\to 2}\frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}\times\frac{\sqrt{3x-2}+\sqrt{x+2}}{\sqrt{3x-2}+\sqrt{x+2}}$$

$$=\lim_{x\to 2}\frac{(x^2-4)(\sqrt{3x-2}+\sqrt{x+2})}{(3x-2-x-2)}$$

$$=\lim_{x\to 2}\frac{(x+2)(x-2)(\sqrt{3x-2}+\sqrt{x+2})}{2(x-2)}$$

$$=\lim_{x\to 2}\frac{(x+2)(\sqrt{3x-2}+\sqrt{x+2})}{2}$$

$$=\frac{(2+2)(2+2)}{2}=\frac{4\times 4}{2}=8$$

**(ii)** Here, $\lim_{x\to 4}\frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}$

$$= \lim_{x\to 4} \frac{3-\sqrt{5+x}}{1-\sqrt{5-x}} \times \frac{3+\sqrt{5+x}}{3+\sqrt{5+x}} \times \frac{1+\sqrt{5-x}}{1+\sqrt{5-x}}$$

$$= \lim_{x\to 4} \frac{(9-5-x)}{(1-5+x)}\left[\frac{1+\sqrt{5-x}}{3+\sqrt{5+x}}\right]$$

$$= \lim_{x\to 4} \frac{-(x-4)}{x-4}\left[\frac{1+\sqrt{5-x}}{3+\sqrt{5+x}}\right]$$

$$= \lim_{x\to 4} \frac{-(1+\sqrt{5-x})}{3+\sqrt{5+x}} = \frac{-(1+1)}{(3+3)} = -\frac{1}{3}$$

**18.** Let $f(x) = \sin(x^2 + 1)$. Then $f(x + h) = \sin\{(x + h)^2 + 1\}$

$$\therefore \quad f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h\to 0} \frac{\sin\{(x+h)^2+1\}-\sin(x^2+1)}{h}$$

$$\Rightarrow \; f'(x) =$$

$$\lim_{h\to 0} \frac{2\sin\left(\frac{2hx+h^2}{2}\right)\cos\left[\frac{(x+h)^2+1+x^2+1}{2}\right]}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h\to 0} \frac{2\sin\left(\frac{2hx+h^2}{2}\right)}{h\left(\frac{2hx+h^2}{2}\right)} \times \left(\frac{2hx+h^2}{2}\right) \times \cos\left\{\frac{(x+h)^2+1+x^2+1}{2}\right\}$$

$$\Rightarrow \quad f'(x) = \lim_{h\to 0} \frac{\sin\left(\frac{2hx+h^2}{2}\right)}{\left(\frac{2hx+h^2}{2}\right)} \times (2x+h) \times \cos\left\{\frac{(x+h)^2+1+x^2+1}{2}\right\}$$

$\Rightarrow \; f'(x) = 1 \times (2x) \times \cos(x^2 + 1) = 2x\cos(x^2 + 1)$

**19.** We have, $y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(1) + \frac{d}{dx}\left(\frac{x}{1!}\right) + \frac{d}{dx}\left(\frac{x^2}{2!}\right) + \frac{d}{dx}\left(\frac{x^3}{3!}\right) + \ldots + \frac{d}{dx}\left(\frac{x^n}{n!}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = 0 + \frac{1}{1!} + \frac{1}{2!}(2x) + \frac{1}{3!}(3x^2) + \ldots + \frac{1}{n!}(nx^{n-1})$$

$$\Rightarrow \quad \frac{dy}{dx} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^{n-1}}{(n-1)!}$$

$$\Rightarrow \quad \frac{dy}{dx} = \left\{1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}\right\} - \frac{x^n}{n!}$$

$$\Rightarrow \quad \frac{dy}{dx} = y - \frac{x^n}{n!} \;\Rightarrow\; \frac{dy}{dx} - y + \frac{x^n}{n!} = 0$$

**20.** We have,

$$\lim_{x\to\frac{\pi}{6}} \frac{2-\sqrt{3}\cos x - \sin x}{(6x-\pi)^2}$$

$$= \lim_{h\to 0} \frac{2-\sqrt{3}\cos\left(\frac{\pi}{6}+h\right) - \sin\left(\frac{\pi}{6}+h\right)}{\left\{6\left(\frac{\pi}{6}+h\right)-\pi\right\}^2} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h\to 0} \frac{2-\sqrt{3}\left(\cos\frac{\pi}{6}\cos h - \sin\frac{\pi}{6}\sin h\right) - \left(\sin\frac{\pi}{6}\cos h + \cos\frac{\pi}{6}\sin h\right)}{36h^2}$$

$$= \lim_{h\to 0} \frac{2 - \frac{3}{2}\cos h + \frac{\sqrt{3}}{2}\sin h - \frac{1}{2}\cos h - \frac{\sqrt{3}}{2}\sin h}{36h^2}$$

$$= \lim_{h\to 0} \frac{2(1-\cos h)}{36h^2}$$

$$= \frac{1}{18}\lim_{h\to 0} \frac{2\sin^2\left(\frac{h}{2}\right)}{h^2} = \frac{1}{9}\lim_{h\to 0}\left(\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right)^2 \times \frac{1}{4}$$

$$= \frac{1}{9} \times (1)^2 \times \frac{1}{4} = \frac{1}{36}$$

## MPP-6 CLASS XI ANSWER KEY

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **1.** | (c) | **2.** | (d) | **3.** | (a) | **4.** | (c) | **5.** | (a) |
| **6.** | (a) | **7.** | (a, c) | **8.** | (a,b,c) | **9.** | (a,d) | **10.** | (a,b) |
| **11.** | (a,c) | **12.** | (a,d) | **13.** | (b,c) | **14.** | (b) | **15.** | (d) |
| **16.** | (c) | **17.** | (6) | **18.** | (1) | **19.** | (2) | **20.** | (2) |

# MPP-6 MONTHLY Practice Problems

Class XI

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer.
Self check table given at the end will help you to check your readiness.

## Limits and Derivatives

**Total Marks : 80** **Time Taken : 60 Min.**

### Only One Option Correct Type

**1.** The value of $\lim\limits_{x\to 1}\dfrac{x^{1/4}-x^{1/5}}{x^3-1}$ is

(a) $\dfrac{1}{20}$ (b) $\dfrac{1}{40}$

(c) $\dfrac{1}{60}$ (d) $\dfrac{3}{20}$

**2.** $\lim\limits_{x\to 0}\dfrac{\sin x - x + \dfrac{x^3}{6}}{x^5}$ equals

(a) $\dfrac{1}{4!}$ (b) $\dfrac{1}{3!}$

(c) $\dfrac{1}{6!}$ (d) $\dfrac{1}{5!}$

**3.** $\lim\limits_{x\to 0}\left(\dfrac{a^x+b^x+c^x+d^x}{4}\right)^{\lambda/x}$, $(a, b, c, d, \lambda > 0)$ is equal to

(a) $abcd$, if $\lambda = 4$ (b) 1, if $\lambda = 1$

(c) $abcd$, if $\lambda = 2$ (d) $(abcd)^{3/4}$, if $\lambda = 1$

**4.** The value of $\lim\limits_{n\to\infty}\dfrac{1+2^4+3^4+\ldots+n^4}{n^5}$

$-\lim\limits_{n\to\infty}\dfrac{1+2^3+3^3+\ldots+n^3}{n^5}$ is

(a) 0 (b) $\dfrac{1}{4}$

(c) $\dfrac{1}{5}$ (d) $\dfrac{1}{30}$

**5.** If $f(1) = 1, f'(1) = 2$ then $\lim\limits_{x\to 1}\dfrac{\sqrt{f(x)}-1}{\sqrt{x}-1}$ equals

(a) 2 (b) 4

(c) 1 (d) $\dfrac{1}{2}$

**6.** If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\infty}}}$, then $\dfrac{dy}{dx} =$

(a) $\dfrac{\cos x}{2y-1}$ (b) $\dfrac{-\cos x}{2y-1}$

(c) $\dfrac{\sin x}{1-2y}$ (d) $\dfrac{-\sin x}{1-2y}$

### One or More Than One Option(s) Correct Type

**7.** If $\lim\limits_{x\to 0}(\cos x + a\sin bx)^{1/x} = e^2$, then

(a) $a = 1, b = 2$ (b) $a = 2, b = 1/2$

(c) $a = 2\sqrt{2}, b = \dfrac{1}{\sqrt{2}}$ (d) $a = 4, b = 2$

**8.** Let $f(x) = \begin{cases} x^2, & x < 1 \\ x, & 1 < x < 4 \\ 4 - x, & x > 4 \end{cases}$, then

(a) $\lim\limits_{x\to1^-} f(x)=1$  (b) $\lim\limits_{x\to1^+} f(x)=1$

(c) $\lim\limits_{x\to4^-} f(x)=4$  (d) $\lim\limits_{x\to4^+} f(x)=4$

**9.** The value of $a$ for which

$$\lim_{x\to0}\frac{(e^x-1)^4}{\sin\left(\dfrac{x^2}{a^2}\right)\log_e\left\{1+\dfrac{x^2}{2}\right\}}=8, \text{ is}$$

(a) $-2$  (b) $-1$

(c) $1$  (d) $2$

**10.** If $f(x)=\left(\dfrac{|x|}{2+|x|}\right)^{2x}$, then

(a) $\lim\limits_{x\to\infty} f(x)=e^{-4}$  (b) $\lim\limits_{x\to-\infty} f(x)=e^{4}$

(c) $\lim\limits_{x\to\infty} f(x)=\infty$  (d) $\lim\limits_{x\to-\infty} f(x)=1$

**11.** Let $L=\lim\limits_{x\to0}\dfrac{a-\sqrt{a^2-x^2}-\dfrac{x^2}{4}}{x^4}$, $a>0$. If $L$ is finite, then

(a) $a=2$  (b) $a=1$

(c) $L=\dfrac{1}{64}$  (d) $L=\dfrac{1}{32}$

**12.** If $a>0$, $b<0$, then $\lim\limits_{x\to0^+}\dfrac{\sqrt{(1-\cos 2ax)}}{\sin bx}$ is equal to

(a) $\dfrac{a\sqrt{2}}{b}$  (b) $-\dfrac{a\sqrt{2}}{b}$

(c) $\dfrac{|a|\sqrt{2}}{|b|}$  (d) $-\dfrac{|a|\sqrt{2}}{|b|}$

**13.** $\lim\limits_{x\to\infty}\sqrt{x}\left(\sqrt{x+1}-\sqrt{x}\right)$ equals

(a) $\lim\limits_{x\to0}\dfrac{\ln(1+x)-x}{x^2}$  (b) $\lim\limits_{x\to0}\dfrac{1-\cos x}{x^2}$

(c) $\lim\limits_{x\to0}\dfrac{\sqrt{(1+x)}-1}{x}$  (d) $\lim\limits_{x\to0}\dfrac{\sqrt{x}}{\sqrt{x+\sqrt{(x^2+2x}}}$

## Comprehension Type

Consider a real valued function $f(x)$, we define a limit as follows $<f(x)>\ =\lim\limits_{h\to0}\dfrac{f^2(x+h)-f^2(x)}{h}$ where $f^2(x)=(f(x))^2$.

**14.** If $u=f(x)$, $v=g(x)$, then the value of $<u\cdot v>$ is

(a) $<u>v+<v>u$  (b) $u^2<v>+v^2<u>$

(c) $<u>+<v>$  (d) $uv<u+v>$

**15.** The value of $<\tan x>$ will be

(a) $\sec^2 x$  (b) $2\sec^2 x$

(c) $\tan x\cdot\sec^2 x$  (d) $2\tan x\cdot\sec^2 x$

## Matrix Match Type

**16.** Match the following :

| | Column I | Column II |
|---|---|---|
| (A) | If the value of $\lim\limits_{x\to1}\dfrac{x+x^2+x^3+\ldots+x^n-n}{x-1}=5050$ then $n=$ | (P) 729 |
| (B) | $\lim\limits_{x\to1}256(\pi-\pi x)\tan\left(\dfrac{\pi x}{2}\right)=$ | (Q) 512 |
| (C) | $\lim\limits_{x\to e}\dfrac{\log x^{729e}-729e}{x-e}=$ | (R) 100 |

| | (A) | (B) | (C) |
|---|---|---|---|
| (a) | P | Q | R |
| (b) | R | P | Q |
| (c) | R | Q | P |
| (d) | P | R | Q |

## Integer Answer Type

**17.** $\lim\limits_{x\to0}\dfrac{x(1+a\cos x)-b\sin x}{x^3}=1$, then $b-3a$ is equal to

**18.** $\lim\limits_{n\to\infty}\left[\dfrac{1}{n^2}+\dfrac{3}{n^2}+\dfrac{5}{n^2}+\ldots..+\dfrac{2n+1}{n^2}\right]$ is equal to

**19.** If $g(x)=e^{2x}\cos 3x$, $g'(0)$ is equal to

**20.** If $f'(0)=2g'(0)$ then derivative of $f(\sin^{-1}x)$ w.r.t. $g(x^2+x)$ at $x=0$ is

◈◈

*Keys are published in this issue. Search now!* ☺

**SELF CHECK**

No. of questions attempted ……

No. of questions correct ……

Marks scored in percentage ……

**Check your score! If your score is**

| | | |
|---|---|---|
| > 90% | EXCELLENT WORK ! | You are well prepared to take the challenge of final exam. |
| 90-75% | GOOD WORK ! | You can score good in the final exam. |
| 74-60% | SATISFACTORY ! | You need to score more next time. |
| < 60% | NOT SATISFACTORY! | Revise thoroughly and strengthen your concepts. |

# CONCEPT BOOSTERS

Class XII

## Probability

This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

### DEFINITIONS

- **Sample space :** The set of all possible outcomes of a trial (random experiment) is called its sample space. It is generally denoted by $S$ and each outcome of the trial is said to be a sample point.
- **Event :** An event is a subset of a sample space.
  - **Simple event :** An event containing only a single sample point.
  - **Compound events :** Events obtained by combining two or more elementary events.
  - **Equally likely events :** Events are equally likely if there is no reason for an event to occur in preference to any other event.
  - **Mutually exclusive or disjoint events :** Events are said to be mutually exclusive or disjoint or incompatible if the occurrence of any one of them prevents the occurrence of all the others.
  - **Mutually non-exclusive events :** Events which are not mutually exclusive are known as compatible events or mutually non exclusive events.
  - **Independent events :** Events are said to be independent if the happening (or non-happening) of one event is not affected by the happening (or non-happening) of others.
  - **Dependent events :** Two or more events are said to be dependent if the happening of one event affects (partially or totally) the other event.
- **Exhaustive number of cases :** The total number of possible outcomes of a random experiment in a trial is known as the exhaustive number of cases.
- **Favourable number of cases :** The number of cases favourable to an event in a trial is the total number of elementary events such that the occurrence of any one of them ensures the happening of the event.
- **Mutually exclusive and exhaustive system of events :** Let $S$ be the sample space associated with a random experiment. Let $A_1, A_2, \ldots, A_n$ be subsets of $S$ such that

  (i) $A_i \cap A_j = \phi$ for all $i \neq j$ and

  (ii) $A_1 \cup A_2 \cup \ldots \cup A_n = S$

  Then the collection of events $A_1, A_2, \ldots, A_n$ is said to form a mutually exclusive and exhaustive system of events.

  In this system, $P(A_1 \cup A_2 \ldots \cup A_n)$
  $= P(A_1) + P(A_2) + \ldots + P(A_n) = 1.$

### PROBABILITY OF AN EVENT

If a random experiment results in $n$ mutually exclusive, equally likely and exhaustive outcomes, out of which $m$ are favourable to the occurrence of an event $A$, then the probability of occurrence of $A$ is given by

$$P(A) = \frac{m}{n} = \frac{\text{Number of outcomes favourable to } A}{\text{Number of total outcomes}}$$

It is obvious that $0 \leq m \leq n$. If an event $A$ is certain to happen, then $m = n$, thus $P(A) = 1$.
If $A$ is impossible to happen, then $m = 0$ and so $P(A) = 0$. Hence, we conclude that $0 \leq P(A) \leq 1$.
Further, if $\bar{A}$ denotes negative of $A$ *i.e.* event that $A$ doesn't happen, then for above cases $m$, $n$; we shall have

$$P(\bar{A}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A)$$

$\therefore\ P(A) + P(\bar{A}) = 1$

## PROBLEMS BASED ON COMBINATION AND PERMUTATION

- **Problems based on combination or selection:** To solve such kind of problems, we use

$${}^nC_r = \frac{n!}{r!(n-r)!}.$$

- **Problems based on permutation or arrangement:** To solve such kind of problems, we use

$${}^nP_r = \frac{n!}{(n-r)!}.$$

## ODDS IN FAVOUR AND ODDS AGAINST AN EVENT

As a result of an experiment if "$a$" of the outcomes are favourable to an event $E$ and "$b$" of the outcomes are against it, then we say that odds are $a$ to $b$ in favour of $E$ or odds are $b$ to $a$ against $E$.

Thus, odds in favour of an event $E$

$$= \frac{\text{Number of favourable cases}}{\text{Number of unfavourable cases}} = \frac{a}{b} = \frac{a/(a+b)}{b/(a+b)} = \frac{P(E)}{P(\bar{E})}.$$

Similarly, odds against an event $E$

$$= \frac{\text{Number of unfavourable cases}}{\text{Number of favourable cases}} = \frac{b}{a} = \frac{P(\bar{E})}{P(E)}.$$

## ADDITION THEOREMS ON PROBABILITY

- **When events are not mutually exclusive :** If $A$ and $B$ are two events which are not mutually exclusive, then

  $P(A \cup B)$ or $P(A + B) = P(A) + P(B) - P(AB)$

  For any three events $A$, $B$, $C$

  $$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

  or $$P(A + B + C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC)$$

- **When events are mutually exclusive :** If $A$ and $B$ are mutually exclusive events, then

  $n(A \cap B) = 0 \ \Rightarrow\ P(A \cap B) = 0$

  $\therefore\ P(A \cup B) = P(A) + P(B)$

  For any three events $A$, $B$, $C$ which are mutually exclusive,

  $$P(A \cap B) = P(B \cap C) = P(C \cap A) = P(A \cap B \cap C) = 0$$

  $\therefore\ P(A \cup B \cup C) = P(A) + P(B) + P(C)$

- **When events are independent :** If $A$ and $B$ are independent events, then $P(A \cap B) = P(A) \cdot P(B)$

  $\therefore\ P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$

## SOME OTHER THEOREMS

- Let $A$ and $B$ be two events associated with a random experiment, then

  (a) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

  (b) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

  If $B \subset A$, then

  (a) $P(A \cap \bar{B}) = P(A) - P(B)$

  (b) $P(B) \le P(A)$

  Similarly, if $A \subset B$, then

  (a) $P(\bar{A} \cap B) = P(B) - P(A)$

  (b) $P(A) \le P(B)$

- Probability of non-occurrence of $A$ or $B$ is

  $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$

- **Generalization of the addition theorem :** If $A_1, A_2, ...., A_n$ are $n$ events associated with a random experiment, then

  $$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{\substack{i,j=1 \\ i \ne j}}^{n} P(A_i \cap A_j) + \sum_{\substack{i,j,k=1 \\ i \ne j \ne k}}^{n} P(A_i \cap A_j \cap A_k) + ... + (-1)^{n-1} P(A_1 \cap A_2 \ .... \ A_n)$$

  If all the events $A_i$ ($i$ = 1, 2, ..., $n$) are mutually exclusive, then

  $$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

  *i.e.* $P(A_1 \cup A_2 ... \cup A_n) = P(A_1) + P(A_2) + ... + P(A_n)$

- **Booley's inequality :** If $A_1, A_2, ..., A_n$ are $n$ events associated with a random experiment, then

  (a) $P\left(\bigcap_{i=1}^{n} A_i\right) \ge \sum_{i=1}^{n} P(A_i) - (n-1)$

  (b) $P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i)$

## CONDITIONAL PROBABILITY

Let $A$ and $B$ be two events associated with a random experiment. Then, the probability of occurrence of $A$ under the condition that $B$ has already occurred and $P(B) \ne 0$, is called the conditional probability and it is denoted by $P(A/B)$.

Thus, $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{n(A \cap B)}{n(B)}$

Similarly, $P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{n(A \cap B)}{n(A)}$

- **Multiplication theorems on probability**
  - If $A$ and $B$ are two events associated with a random experiment, then

$P(A \cap B) = P(A) \cdot P(B/A)$, if $P(A) \neq 0$

or $P(A \cap B) = P(B) \cdot P(A/B)$, if $P(B) \neq 0$.

- **Extension of multiplication theorem :** If $A_1$, $A_2$, ....., $A_n$ are $n$ events related to a random experiment, then

$$P(A_1 \cap A_2 \cap A_3 \cap .... \cap A_n)$$
$$= P(A_1)P(A_2 / A_1)\, P(A_3 / A_1 \cap A_2) \\ ....P(A_n / A_1 \cap A_2 \cap ... \cap A_{n-1})$$

where $P(A_i / A_1 \cap A_2 \cap ... \cap A_{i-1})$ represents the conditional probability of the event $A_i$, given that the events $A_1$, $A_2$, ..., $A_{i-1}$ have already happened.

- **Multiplication theorem for independent events :** If $A$ and $B$ are independent events associated with a random experiment, then $P(A \cap B) = P(A) \cdot P(B)$ *i.e.*, the probability of simultaneous occurrence of two independent events is equal to the product of their probabilities.
- **Extension of multiplication theorem for independent events :** If $A_1$, $A_2$, ..., $A_n$ are independent events associated with a random experiment, then

$$P(A_1 \cap A_2 \cap A_3 \cap .... \cap A_n) = P(A_1)P(A_2) \, ... \, P(A_n)$$

- **Probability of at least one of the $n$ independent events :** If $p_1$, $p_2$, $p_3$, ..., $p_n$ be the probabilities of happening of $n$ independent events $A_1, A_2, A_3, ..., A_n$ respectively, then
  - Probability of happening none of them

$$= P(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 ..... \cap \bar{A}_n)$$
$$= P(\bar{A}_1) \cdot P(\bar{A}_2) \cdot P(\bar{A}_3) ..... P(\bar{A}_n)$$
$$= (1 - p_1)(1 - p_2)(1 - p_3) .... (1 - p_n)$$

  - Probability of happening at least one of them

$$= P(A_1 \cup A_2 \cup A_3 .... \cup A_n)$$
$$= 1 - P(\bar{A}_1)P(\bar{A}_2)P(\bar{A}_3) .... P(\bar{A}_n)$$
$$= 1 - (1 - p_1)(1 - p_2)(1 - p_3) ... (1 - p_n)$$

  - Probability of happening of first event and not happening of the remaining

$$= P(A_1)P(\bar{A}_2)P(\bar{A}_3) ..... P(\bar{A}_n)$$
$$= p_1(1 - p_2)(1 - p_3) ....... (1 - p_n)$$

## TOTAL PROBABILITY AND BAYE'S THEOREM

- **Law of total probability :** Let $S$ be the sample space and let $E_1, E_2, ...., E_n$ be $n$ mutually exclusive and exhaustive events associated with a random experiment. If $A$ is any event which occurs with $E_1$ or $E_2$ or ... or $E_n$, then

$$P(A) = P(E_1)\,P(A / E_1) + P(E_2)\,P(A / E_2) + \\ ... + P(E_n)\,P(A / E_n)$$

- **Baye's theorem :** Let $S$ be a sample space and $E_1$, $E_2$, ..., $E_n$ be $n$ mutually exclusive events such that $\bigcup_{i=1}^{n} E_i = S$ and $P(E_i) > 0$ for $i = 1, 2, ...., n$. We can think of ($E_i$'s as the causes that lead to the outcome of an experiment. The probabilities $P(E_i)$, $i = 1, 2, ...., n$ are called prior probabilities. Suppose the experiment results in an outcome of event $A$, where $P(A) > 0$. We have to find the probability that the observed event $A$ was due to cause $E_i$, that is, we seek the conditional probability $P(E_i/A)$. These probabilities are called posterior probabilities, given by Baye's rule as

$$P(E_i / A) = \frac{P(E_i) \cdot P(A / E_i)}{\sum_{k=1}^{n} P(E_k)\, P(A / E_k)}$$

## BINOMIAL DISTRIBUTION

- **Geometrical method for probability :** When the number of points in the sample space is infinite, it becomes difficult to apply classical definition of probability. For instance, if we are interested to find the probability that a point selected at random from the interval [1, 6] lies either in the interval [1, 2] or [5, 6], we cannot apply the classical definition of probability. In this case we define the probability as follows:

$$P\{x \in A\} = \frac{\text{Measure of region } A}{\text{Measure of the sample space } S}$$

where measure stands for length, area or volume depending upon whether $S$ is a one-dimensional, two-dimensional or three-dimensional region.

- **Probability distribution :** Let $S$ be a sample space. A random variable $X$ is a function from the set $S$ to $R$, the set of real numbers.
  If $X$ is a random variable defined on the sample space $S$ and $r$ is a real number, then $\{X = r\}$ is an event.
  Let $(X = x_i)$ is an event, we can talk of $P(X = x_i)$. If $P(X = x_i) = P_i(1 \leq i \leq n)$, then the system of numbers $\begin{pmatrix} x_1 & x_2 & .... & x_n \\ p_1 & p_2 & .... & p_n \end{pmatrix}$ is said to

be the probability distribution of the random variable $X$.

The expectation (mean) of the random variable $X$ is defined as $E(X)=\sum_{i=1}^{n} p_i x_i$ and the variance of $X$ is defined as $\mathrm{Var}(X)=\sum_{i=1}^{n} p_i(x_i-E(X))^2$

$$=\sum_{i=1}^{n} p_i x_i^2-(E(X))^2$$

- **Binomial probability distribution :** A random variable $X$ which takes values 0, 1, 2, ..., $n$ is said to follow binomial distribution if its probability distribution function is given by $P(X=r)={}^nC_r p^r q^{n-r}$, $r=0,1,2, \ldots, n$ where $p$, $q > 0$ such that $p + q = 1$.
  The notation $X \sim B(n, p)$ is generally used to denote that the random variable $X$ follows binomial distribution with parameters $n$ and $p$.
- **Mean and variance of the binomial distribution :** The binomial probability distribution is

| $X$ | 0 | 1 | 2 | ..... | $n$ |
|---|---|---|---|---|---|
| $P(X)$ | ${}^nC_0 q^n p^0$ | ${}^nC_1 q^{n-1} p$ | ${}^nC_2 q^{n-2} p^2$ | ..... | ${}^nC_n q^0 p^n$ |

  The mean of the distribution is
  $$\sum_{i=1}^{n} X_i p_i = \sum_{X=1}^{n} X \cdot {}^nC_X q^{n-X} p^X = np,$$
  The variance of the binomial distribution is $\sigma^2 = npq$ and the standard deviation is $\sigma=\sqrt{(npq)}$
- **Use of multinomial expansion :** If a die has $m$ faces marked with the numbers 1, 2, 3, ...., $m$ and if such $n$ dice are thrown, then the probability that the sum of the numbers exhibited on the upper faces equal to $p$ is given by the coefficient of $x^p$ in the expansion of $\frac{(x+x^2+x^3+\ldots+x^m)^n}{m^n}$.
- **The poisson distribution :** Let $X$ be a discrete random variable which can take on the values 0, 1, 2,... such that the probability function of $X$ is given by
  $$f(x)=P(X=x)=\frac{\lambda^x e^{-\lambda}}{x!},\ x=0, 1, 2, \ldots$$
  where $\lambda$ is a given positive constant. This distribution is called the Poisson distribution and a random variable having this distribution is said to be Poisson distributed.

## SALIENT POINTS

- Independent events are always taken from different experiments, while mutually exclusive events are taken from a single experiment.
- Independent events can happen together while mutually exclusive events cannot happen together.
- Independent events are connected by the word "and" but mutually exclusive events are connected by the word "or".
- Number of exhaustive cases of tossing $n$ coins simultaneously (or of tossing a coin $n$ times) $= 2^n$.
- Number of exhaustive cases of throwing $n$ dice simultaneously (or throwing one dice $n$ times) $= 6^n$.
- **Probability regarding $n$ letters and their envelopes:** If $n$ letters corresponding to $n$ envelopes are placed in the envelopes at random, then
  (i) Probability that all letters are in right envelopes $= 1/n!$.
  (ii) Probability that all letters are not in right envelopes $=1-\frac{1}{n!}$.
  (iii) Probability that no letter is in right envelopes
  $$=\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\ldots+(-1)^n\frac{1}{n!}.$$
  (iv) Probability that exactly $r$ letters are in right envelopes
  $$=\frac{1}{r!}\left[\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\ldots\ldots+(-1)^{n-r}\frac{1}{(n-r)!}\right].$$
- If odds in favour of an event are $a : b$, then the probability of the occurrence of that event is $\frac{a}{a+b}$ and the probability of non-occurrence of that event is $\frac{b}{a+b}$.
- If odds against an event are $a : b$, then the probability of the occurrence of that event is $\frac{b}{a+b}$ and the probability of non-occurrence of that event is $\frac{a}{a+b}$.

## PROBLEMS

### Single Correct Answer Type

**1.** There are $n$ letters and $n$ addressed envelopes. The probability that all the letters are not kept in the right envelope, is

(a) $\frac{1}{n!}$ (b) $1-\frac{1}{n!}$

(c) $1-\frac{1}{n}$ (d) None of these

**2.** If a dice is thrown twice, then the probability of getting 1 in the first throw only is

(a) $\frac{1}{36}$ (b) $\frac{3}{36}$ (c) $\frac{5}{36}$ (d) $\frac{1}{6}$

**3.** Two dice are thrown simultaneously. The probability of getting the sum 2 or 8 or 12 is

(a) $\frac{5}{18}$ (b) $\frac{7}{36}$ (c) $\frac{7}{18}$ (d) $\frac{5}{36}$

**4.** One card is drawn from each of two ordinary packs of 52 cards. The probability that at least one of them is an ace of heart, is

(a) $\frac{103}{2704}$ (b) $\frac{1}{2704}$ (c) $\frac{2}{52}$ (d) $\frac{2601}{2704}$

**5.** The probability of happening an event $A$ in one trial is 0.4. The probability that the event $A$ happens at least once in three independent trials is

(a) 0.936 (b) 0.784 (c) 0.904 (d) 0.216

**6.** A problem of mathematics is given to three students whose chances of solving the problem are 1/3, 1/4 and 1/5 respectively. The probability that the question will be solved is

(a) 2/3 (b) 3/4 (c) 4/5 (d) 3/5

**7.** A card is drawn from a well shuffled pack of cards. The probability of getting a queen of club or king of heart is

(a) $\frac{1}{52}$ (b) $\frac{1}{26}$

(c) $\frac{1}{18}$ (d) None of these

**8.** Two dice are thrown simultaneously. What is the probability of obtaining sum of the numbers less than 11

(a) $\frac{17}{18}$ (b) $\frac{1}{12}$

(c) $\frac{11}{12}$ (d) None of these

**9.** A man and a woman appear in an interview for two vacancies in the same post. The probability of man's selection is 1/4 and that of the woman's selection is 1/3. What is the probability that none of them will be selected?

(a) 1/2 (b) 1/12

(c) 1/4 (d) None of these

**10.** A number is chosen from first 100 natural numbers. The probability that the number is even or divisible by 5, is

(a) 3/4 (b) 2/3 (c) 4/5 (d) 3/5

**11.** A determinant is chosen at random from the set of all determinants of order 2 with elements 0 or 1 only. The probability that the determinant chosen is non-zero is

(a) 3/16 (b) 3/8

(c) 1/4 (d) None of these

**12.** A bag contains 19 tickets numbered from 1 to 19. A ticket is drawn and then another ticket is drawn without replacement. The probability that both the tickets will show even number, is

(a) $\frac{9}{19}$ (b) $\frac{8}{18}$ (c) $\frac{9}{18}$ (d) $\frac{4}{19}$

**13.** A bag contains 3 white, 3 black and 2 red balls. One by one three balls are drawn without replacing them. The probability that the third ball is red, is

(a) 1/2 (b) 1/3 (c) 2/3 (d) 1/4

**14.** The probability of India winning a test match against West Indies is $\frac{1}{2}$. Assuming independence from match to match, the probability that in a 5 match series India's second win occurs at the third test, is

(a) 2/3 (b) 1/2 (c) 1/4 (d) 1/8

**15.** Seven chits are numbered from 1 to 7. Three are drawn one by one with replacement. The probability that the least number on any selected chit is 5, is

(a) $1-\left(\frac{2}{7}\right)^4$ (b) $4\left(\frac{2}{7}\right)^4$

(c) $\left(\frac{3}{7}\right)^3$ (d) None of these

**16.** A bag contains 3 black and 4 white balls. Two balls are drawn one by one at random without replacement. The probability that the second drawn ball is white, is

(a) $\frac{4}{49}$ (b) $\frac{1}{7}$ (c) $\frac{4}{7}$ (d) $\frac{12}{49}$

**17.** A dice is rolled three times, the probability of getting a larger number than the previous number each time is

(a) $\frac{15}{216}$ (b) $\frac{5}{54}$ (c) $\frac{13}{216}$ (d) $\frac{1}{18}$

**18.** The probabilities of a student getting I, II and III division in an examination are respectively $\frac{1}{10}, \frac{3}{5}$ and $\frac{1}{4}$. The probability that the student fails in the examination is

(a) $\frac{197}{200}$ (b) $\frac{27}{100}$

(c) $\frac{83}{100}$ (d) None of these

**19.** A fair coin is tossed repeatedly. If tail appears on first four tosses then the probability of head appearing on fifth toss equals

(a) $\frac{1}{2}$ (b) $\frac{1}{32}$ (c) $\frac{31}{32}$ (d) $\frac{1}{5}$

**20.** In a college, 25% of the boys and 10% of the girls offer Mathematics. The girls constitute 60% of the total number of students. If a student is selected at random and is found to be studying Mathematics, the probability that the student is a girl, is

(a) 1/6 (b) 3/8 (c) 5/8 (d) 5/6

**21.** A bag contains 3 red and 5 black balls and a second bag contains 6 red and 4 black balls. A ball is drawn from each bag. The probability that one is red and other is black, is

(a) 3/20 (b) 21/40

(c) 3/8 (d) None of these

**22.** If $\frac{1+3p}{3}, \frac{1-p}{2}$ and $\frac{1-2p}{2}$ are the probabilities of three mutually exclusive events, then the set of all values of $p$ is

(a) $\phi$ (b) $\left[\frac{1}{2}, \frac{1}{3}\right]$ (c) [0, 1] (d) $\left[\frac{1}{3}, \frac{2}{3}\right]$

**23.** The probability that an anti aircraft gun can hit an enemy plane at the first, second and third shot are 0.6, 0.7 and 0.1 respectively. The probability that the gun hits the plane is

(a) 0.108 (b) 0.892 (c) 0.14 (d) 0.91

**24.** A team of 8 couples (husband and wife), attend a lucky draw in which 4 persons picked up for a prize. The probability that there is at least one couple is

(a) 11/39 (b) 15/39 (c) 14/39 (d) 12/39

**25.** If $A$ and $B$ are two events such that $P(A) > 0$ and $P(B) \neq 1$, then $P(\bar{A}/\bar{B})$ is equal to

(a) $1 - P(A/B)$ (b) $1 - P(\bar{A}/\bar{B})$

(c) $\frac{P(\bar{A})}{P(\bar{B})}$ (d) $\frac{1-P(A \cup B)}{P(\bar{B})}$

## Multiple Correct Answer Type

**26.** The probability that a 50 year-old man will be alive at 60 is 0.83 and the probability that a 45 year-old woman will be alive at 55 is 0.87. Then

(a) The probability that both will be alive for the next 10 years is 0.7221

(b) At least one of them will alive for the next 10 years is 0.9779

(c) At least one of them will alive for the next 10 years is 0.8230

(d) The probability that both will be alive for the next 10 years is 0.6320

**27.** Each of the $n$ bags contains $a$ white and $b$ black balls. One ball is transferred from first bag to the second bag then one ball is transferred from second bag to the third bag and so on. Let $p_n$ be the probability that ball transferred from $n^{th}$ bag is white, then

(a) $p_1 = \frac{a}{a+b}$ (b) $p_2 = \frac{a}{a+b}$

(c) $p_3 = \frac{a}{a+b}$ (d) $p_4 = \frac{a}{a+b}$

**28.** $A$, $B$ are two events of a random experiment such that $P(\bar{A}) = 0.3$, $P(B) = 0.4$ and $P(A \cap \bar{B}) = 0.5$. Then

(a) $P(A \cup B) = 0.9$ (b) $P(B \cap \bar{A}) = 0.2$

(c) $P(\bar{A} \cup \bar{B}) = 0.8$ (d) $P(B / A \cup \bar{B}) = 0.25$

**29.** Given that $x \in [0, 1]$ and $y \in [0, 1]$. Let $A$ be the event of $(x, y)$ satisfying $y^2 \leq x$ and $B$ be the event of $(x, y)$ satisfying $x^2 \leq y$. Then

(a) $P(A \cap B) = \frac{1}{3}$

(b) $A$, $B$ are not exhaustive

(c) $A$, $B$ are mutually exclusive

(d) $A$, $B$ are not independent.

**30.** A random variable $x$ takes values 0, 1, 2, 3, ... with probability proportions to $(x+1)\left(\frac{1}{5}\right)^x$ then

(a) $P(X=0) = \frac{16}{25}$ (b) $P(X \leq 1) = \frac{112}{125}$

(c) $P(X \geq 1) = \frac{9}{25}$ (d) None of these

**31.** The probabilities that a student passes in mathematics, physics and chemistry are $m$, $p$ and $c$ respectively. Of these subjects, a student has a 75% chance of passing in at least one, a 50% chance of passing in at least two and a 40% chance of passing in exactly two subjects. Which of the following relations are true?

(a) $p+m+c=\frac{19}{20}$ (b) $p+m+c=\frac{27}{20}$

(c) $pmc=\frac{1}{10}$ (d) $pmc=\frac{1}{4}$

**32.** A boy has a collection of blue and green marbles. The number of blue marbles belong to the set $\{2, 3, 4, ...., 13\}$. If two marbles are chosen simultaneously and at random from his collection, then the probability that they have different colours is 1/2. Possible number of blue marbles is

(a) 3 (b) 6 (c) 10 (d) 12

**33.** $P(A \cap B) = P(A \cup B)$ if the relation between $P(A)$ and $P(B)$ is .........

(a) $P(A) = P(B)$ (b) $P(A) = P(A \cap B)$

(c) $P(A) = P(A \cup B)$ (d) $P(B) = P(A \cap B)$

**34.** A consignment of 15 record players contains 4 defectives. The record players are selected at random, one by one and examined. The one examined are not put back. Then

(a) Probability of getting exactly 3 defectives in the examination of 8 record players is $\frac{{}^4C_3 \times {}^{11}C_5}{{}^{15}C_8}$

(b) Probability that $9^{th}$ one examined is the last defective is $\frac{8}{195}$

(c) Probability that $9^{th}$ examined record player is defective given that there were 3 defectives in the first 8 players examined is 1/7

(d) Probability that $9^{th}$ one examined is last defective is $\frac{8}{197}$

**35.** Which of the following statements are true?

(a) The probability that birthday of twelve people will fall in 12 calendar months $=\frac{12!}{(12)^{12}}$

(b) The probability that birthday of six people will fall in exactly two calendar months $=\frac{{}^{12}C_2(2^6-2)}{(12)^6}$

(c) The probability that birthday of six people will fall is exactly two calendar months $=\frac{{}^{12}C_3(2^7-2)}{(12)^7}$

(d) The probability that birthdays of $n$ ($n \le 365$) people are different $=\frac{{}^{365}P_n}{(365)^n}$

**36.** If $A$ and $B$ are two events. The probability that at most one of $A$, $B$ occurs is

(a) $1 - P(A \cap B)$ (b) $P(\bar{A})+P(\bar{B})-P(\bar{A}\cap\bar{B})$

(c) $P(\bar{A})+P(\bar{B})+P(A\cup B)-1$

(d) $P(A\cap\bar{B})+P(\bar{A}\cap B)+P(\bar{A}\cap\bar{B})$

**37.** If $P(A) = 3/5$ and $P(B) = 2/3$ then

(a) $P(A\cup B)\ge\frac{2}{3}$ (b) $\frac{4}{15}\le P(A\cap B)\le\frac{3}{5}$

(c) $P(A\cap\bar{B})\le\frac{1}{3}$ (d) $P(A\cup B)\ge\frac{3}{5}$

**38.** When a coin is flipped '$n$' times and the probability that the first head comes after exactly $m$ ($n > m + 1$) tails is $\frac{1}{2^6}$ then

(a) $n = 8, m = 5$ (b) $n = 7, m = 5$

(c) $n = 8, m = 6$ (d) $n = 5, m = 3$

**39.** Let $E$, $F$ be two independent events .The probability that both $E$ and $F$ happen is $\frac{1}{12}$ and probability that neither $E$ nor $F$ happens is 1/2, then

(a) $3P(E) = 4P(F) = 1$ (b) $P(E\cup F)=\frac{1}{2}$

(c) $4P(E) = 3P(F) = 1$ (d) $P(E) = P(F)$

**40.** When a fair die is thrown twice, let $(a, b)$ denote the outcome in which the first throw shows $a$ and the second shows $b$. Further, let $A$, $B$ and $C$ be the following events : $A = \{(a, b) \mid a \text{ is odd}\}$, $B = \{(a, b) \mid b \text{ is odd}\}$ and $C = \{(a, b) \mid a + b \text{ is odd}\}$. Then

(a) $P(A \cap B) = 1/4$ (b) $P(B \cap C) = 1/4$

(c) $P(A \cap C) = 1/4$ (d) $P(A \cap B \cap C) = 0$

**41.** A certain coin is tossed with probability of showing head being $p$. Let $Q$ denotes the probability that when the coin is tossed four times, the number of heads obtained is even. Then

(a) there is no value of $p$, if $Q=\frac{1}{4}$

(b) there is exactly one value of $p$, if $Q=\frac{3}{4}$

(c) there are exactly two values of $p$, if $Q = \frac{3}{5}$

(d) there are exactly four values of $p$, if $Q = \frac{4}{5}$

**42.** If $A$ and $B$ are two events such that $P(B) \neq 1$, $B^C$ denotes the event complementary to $B$, then

(a) $P(A/B^C) = \dfrac{P(A) - P(A \cap B)}{1 - P(B)}$

(b) $P(A \cap B) \geq P(A) + P(B) - 1$

(c) $P(A) \cdot P(B) > P(A \cap B)$

(d) None of these

## Comprehension Type

### Paragraph for Q. No. 43 to 45

A box contains $n$ coins. Let $P(E_i)$ be the probability that exactly $i$ out of $n$ coins are biased. If $P(E_i)$ is directly proportional to $i(i + 1)$; $1 \leq i \leq n$.

**43.** Proportionality constant $K$ is equal to

(a) $\dfrac{3}{n(n^2+1)}$ (b) $\dfrac{1}{(n^2+1)(n+2)}$

(c) $\dfrac{3}{n(n+1)(n+2)}$ (d) $\dfrac{1}{(n+1)(n+2)(n+3)}$

**44.** If $P$ be the probability that a coin selected at random is biased then $\lim\limits_{n\to\infty} P$ is

(a) 1/4 (b) 3/4 (c) 3/5 (d) 7/8

**45.** If a coin selected at random is found to be biased then the probability that it is the only biased coin in the box is

(a) $\dfrac{1}{(n+1)(n+2)(n+3)(n+4)}$

(b) $\dfrac{12}{n(n+1)(n+2)(3n+1)}$

(c) $\dfrac{24}{n(n+1)(n+2)(2n+1)}$

(d) $\dfrac{24}{n(n+1)(n+2)(3n+1)}$

### Paragraph for Q. No. 46 to 48

A slip of paper is given to a person $A$ who marks it either with a plus sign or a minus sign. The probability of his writing a plus sign is 1/3. $A$ passes the slip to $B$, who may either leave it alone or change the sign before passing it to $C$. Next $C$ passes the slip to $D$ after perhaps changing the sign. Finally $D$ passes it to a referee after perhaps changing the sign. $B$, $C$, $D$ each change the sign with probability 2/3.

**46.** The probability that the referee observes a plus sign on the slip if it is known that $A$ wrote a plus sign is

(a) 14/27 (b) 16/27 (c) 13/27 (d) 17/27

**47.** The probability that the referee observes a plus sign on the slip if it is known that $A$ wrote a minus sign is

(a) 16/27 (b) 14/27 (c) 13/27 (d) 11/27

**48.** If the referee observes a plus sign on the slip then the probability that $A$ originally wrote a plus sign is

(a) 13/41 (b) 19/27 (c) 17/25 (d) 21/37

### Paragraph for Question No. 49 to 51

A commander of an army battalion is punishing two of his soldiers $X$ and $Y$. He arranged a duel between them. The rules of the duel are that they are to pick up their guns and shoot at each other simultaneously.
If one or both hit, then the duel is over. If both shot miss then they repeat the process. Suppose that the results of the shots are independent and that each shot of $X$ will hit $Y$ with probability 0.4 and each shot of $Y$ will hit $X$ with probability 0.2. Now answer the following questions.

**49.** The probability that the duel ends after first round is

(a) 11/25 (b) 12/25 (c) 13/25 (d) 2/5

**50.** The probability that $X$ is not hit, is

(a) 3/25 (b) 7/25 (c) 5/13 (d) 8/13

**51.** The probability that both the soldiers are hit, is

(a) 5/13 (b) 2/13 (c) 8/13 (d) 1/13

## Matrix – Match Type

**52.** $A$ is a set containing $n$ elements. A subset $P$ of $A$ is chosen at random. The set $A$ is reconstructed by replacing the elements of the subset $P$. A subset $Q$ of $A$ is again chosen at random. The probability that

| Column-I | | Column-II | |
|---|---|---|---|
| (A) | $P \cap Q = \phi$ | (p) | $n(3^{n-1})/4^n$ |
| (B) | $P \cap Q$ is a singleton | (q) | $(3/4)^n$ |
| (C) | $P \cap Q$ contains 2 elements | (r) | $^{20}C_n/4^n$ |
| (D) | $\|P\| = \|Q\|$, where $\|X\|$ = number of elements in $X$ | (s) | $3^{n-2}(n(n-1))/2(4^n)$ |

**53.** Two dice are thrown. Let $A$ be the event that sum of the points on the two dice is odd and $B$ be the event that atleast one 3 is there, then match the following :

| Column-I | | Column-II | |
|---|---|---|---|
| (A) | $P(A \cup B)$ | (p) | 12/36 |
| (B) | $P(A \cap B)$ | (q) | 6/36 |
| (C) | $P(A \cap \bar{B})$ | (r) | 23/36 |
| (D) | $P(B)$ | (s) | 11/36 |

## Integer Answer Type

**54.** An experiment has 12 equally likely outcomes. Let $A$ and $B$ be non-empty events of the experiment. If $A$ consists of 8 outcomes then the possible number of outcomes that $B$ must have so that $A$ and $B$ are independent is

**55.** In a group of people, if 4 are selected at a random, the probability that any two of the four do not have same month of birth is $p$, then $\frac{96p}{11}$ is equal to

**56.** Functions are formed from set $A = \{1, 2, 3\}$ to set $B = \{1, 2, 3, 4, 5\}$ and one function is selected at random. If $P$ is the probability that function satisfying $f(i) \le f(j)$ whenever $i < j$, then value of $25P$ is equal to

**57.** If the sides of triangle are decided by throwing a die thrice, the probability that the triangle is isosceles or equilateral is $1/k$ then $k =$

**58.** If $p$, $q$ are chosen randomly with replacement from the set {1, 2, 3, ..., 10}, the probability that the roots of the equation $x^2 + px + q = 0$ are real, is $\frac{k^3+23}{50}$ then $k =$

**59.** An unbiased coin is tossed 12 times .The probability that at least 7 consecutive heads show up is $\frac{K}{256}$, then $K =$

**60.** An unbiased coin is tossed. If the result is a head, a pair of unbiased dice is rolled and the number obtained by adding the numbers on the two faces is noted. If the result is a tail, a card from a well shuffled pack of eleven cards numbered 2, 3, 3, 4, .... 12 is picked and the number on the card is noted. The probability that the noted number is either 7 or 8 is $\frac{P}{792}$, then the digit in tens place of $P$ is _______.

## SOLUTIONS

**1. (b) :** Required probability = 1 – $P$ (All letters in right envelope) $= 1 - \frac{1}{n!}$

**2. (c) :** Probability of getting 1 in first throw $= \frac{1}{6}$

Probability of not getting 1 in second throw $= \frac{5}{6}$

Both are independent events, so required probability $= \frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$.

**3. (b) :** The sum 2 can be found in one way *i.e.* (1, 1) The sum 8 can be found in five ways *i.e.* (6, 2), (5, 3), (4, 4), (3, 5), (2, 6). Similarly the sum 12 can be found in one way *i.e.*, (6, 6).

Hence, required probability = 7/36.

**4. (a) :** Required probability = 1 – $P$ (no ace of heart)

$= 1 - \frac{51}{52} \cdot \frac{51}{52} = \frac{(52+51)}{52 \cdot 52} = \frac{103}{2704}$.

**5. (b) :** Here, $P(A) = 0.4$ and $P(\bar{A}) = 0.6$

Probability that $A$ does not happen at all = $(0.6)^3$

Thus, required probability = 1 – $(0.6)^3$ = 0.784

**6. (d) :** The probability of students not solving the problem are $1 - \frac{1}{3} = \frac{2}{3}, 1 - \frac{1}{4} = \frac{3}{4}$ and $1 - \frac{1}{5} = \frac{4}{5}$

Therefore, the probability that the problem is not solved by any one of them $= \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} = \frac{2}{5}$

Hence, the probability that problem is solved $= 1 - \frac{2}{5} = \frac{3}{5}$.

**7. (b) :** The probability of card to be queen of club is $\frac{1}{52}$ and probability of card to be a king of heart is also $\frac{1}{52}$.

Both are mutually exclusive events, hence the required probability $= \frac{1}{52} + \frac{1}{52} = \frac{2}{52} = \frac{1}{26}$.

**8. (c) :** Favourable cases to get the sum not less than 11 are {(5, 6), (6, 6), (6, 5)} = 3

Hence favourable cases to get the sum less than 11 are (36 – 3) = 33. So, required probability $= \frac{33}{36} = \frac{11}{12}$.

**9. (a) :** Let $E_1$ be the event that man will be selected and $E_2$ the event that woman will be selected. Then

$$P(E_1)=\frac{1}{4} \Rightarrow P(\bar{E}_1)=1-\frac{1}{4}=\frac{3}{4} \text{ and}$$

$$P(E_2)=\frac{1}{3} \quad \Rightarrow \quad P(\bar{E}_2)=\frac{2}{3}$$

Clearly $E_1$ and $E_2$ are independent events.

So, $P(\bar{E}_1 \cap \bar{E}_2)=P(\bar{E}_1)\times P(\bar{E}_2)$

$$=\frac{3}{4}\times\frac{2}{3}=\frac{1}{2}$$

**10. (d) :** Let $E_1$ be event that the number is even and $E_2$ be the event that the number is odd and divisible by 5.

$$\Rightarrow \quad P(E_1)=\frac{50}{100} \text{ and } P(E_2)=\frac{10}{100}$$

Hence, required probability $=\dfrac{50+10}{100}=\dfrac{3}{5}$.

**11. (b) :** A determinant of order 2 is of the form $\Delta=\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

It is equal to $ad - bc$. The total number of ways of choosing $a$, $b$, $c$ and $d$ is $2 \times 2 \times 2 \times 2 = 16$. Now $\Delta \neq 0$ if and only if either $ad = 1$, $bc = 0$ or $ad = 0$, $bc = 1$. But $ad = 1$, $bc = 0$ iff $a = d = 1$ and one of $b$, $c$ is zero. Therefore $ad = 1$, $bc = 0$ in three cases, similarly $ad = 0$, $bc = 1$ in three cases. Therefore the required probability $=\dfrac{6}{16}=\dfrac{3}{8}$.

**12. (d) :** The probability of getting an even number in first draw $=\dfrac{9}{19}$. The probability of getting an even number in second draw $=\dfrac{8}{18}$. Both are independent event and so required probability $=\dfrac{9}{19}\times\dfrac{8}{18}=\dfrac{4}{19}$.

**13. (d) :** Let $R$ stand for drawing red ball, $B$ for drawing black ball and $W$ for drawing white ball.

Then required probability

$= P(WWR) + P(BBR) + P(WBR) + P(BWR)$
$\quad + P(WRR) + P(BRR) + P(RWR) + P(RBR)$

$$=\frac{3\cdot2\cdot2}{8\cdot7\cdot6}+\frac{3\cdot2\cdot2}{8\cdot7\cdot6}+\frac{3\cdot3\cdot2}{8\cdot7\cdot6}+\frac{3\cdot3\cdot2}{8\cdot7\cdot6}+\frac{3\cdot2\cdot1}{8\cdot7\cdot6}+\frac{3\cdot2\cdot1}{8\cdot7\cdot6}+\frac{2\cdot3\cdot1}{8\cdot7\cdot6}+\frac{2\cdot3\cdot1}{8\cdot7\cdot6}$$

$$=\frac{2}{56}+\frac{2}{56}+\frac{3}{56}+\frac{3}{56}+\frac{1}{56}+\frac{1}{56}+\frac{1}{56}+\frac{1}{56}=\frac{1}{4}.$$

**14. (c) :** The sample space is $[LWW, WLW]$

$\therefore \quad P(LWW) + P(WLW)$

= Probability that in 5 match series, it is India's second win

$$=P(L)P(W)P(W)+P(W)P(L)P(W)=\frac{1}{8}+\frac{1}{8}=\frac{2}{8}=\frac{1}{4}.$$

**15. (c) :** $P$(5 or 6 or 7) in one draw $=\dfrac{3}{7}$

$\therefore$ Probability that in each of 3 draws, the chits bear 5 or 6 or 7 $=\left(\dfrac{3}{7}\right)^3$

**16. (c) :** Second white ball can draw in two ways.

(i) First is white and second is white

$\Rightarrow$ Probability $=\dfrac{4}{7}\times\dfrac{3}{6}=\dfrac{2}{7}$

(ii) First is black and second is white

$\Rightarrow$ Probability $=\dfrac{3}{7}\times\dfrac{4}{6}=\dfrac{2}{7}$

Hence, required probability $=\dfrac{2}{7}+\dfrac{2}{7}=\dfrac{4}{7}$.

**17. (b) :** Exhaustive number of cases $= 6^3 = 216$

Obviously, the second number has to be greater than unity. If the second number is $i$ ($i > 1$), then the first can be chosen in $i - 1$ ways and the third in $6 - i$ ways and hence three numbers can be chosen in $(i - 1) \times 1 \times (6 - i)$ ways. But the second number can be 2, 3, 4, 5. Thus the favourable number of cases

$$=\sum_{i=2}^{5}(i-1)(6-i)=1\times4+2\times3+3\times2+4\times1=20$$

Hence, required probability $=\dfrac{20}{216}=\dfrac{5}{54}$.

**18. (d) :** $A$ denote the event getting I;
$B$ denote the event getting II;
$C$ denote the event getting III;
and $D$ denote the event getting fail.
Obviously, these four event are mutually exclusive and exhaustive, therefore $P(A) + P(B) + P(C) + P(D) = 1$
$\Rightarrow P(D) = 1 - 0.95 = 0.05$.

**19. (a) :** Appearance of head on fifth toss does not depend on the outcomes of first four tosses. Hence

$$P(\text{head on } 5^{\text{th}} \text{ toss})=\frac{1}{2}.$$

**20. (b) :** Let 100 students studying in which 60% girls and 40% boys.

$\Rightarrow$ Boys = 40, Girls = 60

Number of boys offer Maths $=\frac{25}{100}\times 40=10$

Number of girls offer Maths $=\frac{10}{100}\times 60=6$

It means, 16 students offer Maths.

$\therefore$ Required probability $=\frac{6}{16}=\frac{3}{8}.$

**21. (b) :** From bag $A$, $P$ (red ball) $= p_1=\frac{3}{8}$

$P$ (black ball) $= p_2=\frac{5}{8}$

From bag $B$, $P$ (red ball) $= p_3=\frac{6}{10}$

$P$ (black ball) $= p_4=\frac{4}{10}$

Required probability
$= P$ [(red ball from bag $A$ and black from $B$) or (red from bag $B$ and black from $A$)]

$$=p_1\times p_4+p_2\times p_3=\frac{3}{8}\times\frac{4}{10}+\frac{5}{8}\times\frac{6}{10}=\frac{21}{40}$$

**22. (a) :** We have

$$0\le\frac{1+3p}{3},\frac{1-p}{2}\text{ and }\frac{1-2p}{2}\le 1\Rightarrow p\in\left[-\frac{1}{3},\frac{1}{2}\right]$$

Further if the events (say $E_1$, $E_2$ and $E_3$) are exclusive, then its necessary and sufficient condition is
$P(E_1\cup E_2\cup E_3)=P(E_1)+P(E_2)+P(E_3)$

$$\Rightarrow\ P(E_1\cup E_2\cup E_3)=\frac{8-3p}{6}$$

$$\Rightarrow 0\le\frac{8-3p}{6}\le 1\Rightarrow p\in\left[\frac{2}{3},\frac{8}{3}\right]$$

Hence, the required set is $\phi$.

**23. (b) :** Let the events of hitting the enemy plane at the first, second and third shot are respectively $A$, $B$ and $C$. Then as given $P(A) = 0.6$, $P(B) = 0.7$, $P(C) = 0.1$
Since events $A$, $B$, $C$ are independent, so required probability $= P(A + B + C)$
$=1-P(\bar{A})P(\bar{B})P(\bar{C})$
$= 1 - (1 - 0.6)(1 - 0.7)(1 - 0.1)$
$= 1 - (0.4)(0.3)(0.9) = 1 - 0.108 = 0.892$

**24. (b) :** Probability of selecting at least one couple for the prize $= 1$ – probability of not selecting any couple for the prize

$$=1-\frac{({}^{16}C_1\times{}^{14}C_1\times{}^{12}C_1\times{}^{10}C_1)4!}{{}^{16}C_4}$$

$$=1-\frac{16\times14\times12\times10}{16\times15\times14\times13}=\frac{15}{39}$$

**25. (d) :**

$$P(\bar{A}/\bar{B})=\frac{P(\bar{A}\cap\bar{B})}{P(\bar{B})}=\frac{P(\overline{A\cap B})}{P(\bar{B})}=\frac{1-P(A\cup B)}{P(\bar{B})}$$

**26. (a, b) :** The probability that both will be alive for 10 years hence *i.e.*, the probability that the man and the woman both be alive 10 years hence is $0.83\times0.87$ $= 0.7221$.
The probability that at least one of them will be alive is 1-$P$(none of them remains alive next 10 years)
$= 1 - (1 - 0.83)(1 - 0.87) = 1 - 0.17\times0.13 = 0.9779$

**27. (a, b, c, d) :** $p_1=\frac{a}{a+b}$ is obvious

$$p_2=\frac{a}{a+b}\cdot\frac{a+1}{a+b+1}+\frac{b}{a+b}\cdot\frac{a}{a+b+1}=\frac{a}{a+b}$$

$\Rightarrow$ Choices (a) and (b) are correct

Let us assume $p_n=\frac{a}{a+b}$ for some $n$ then,

$$p_{n+1}=p_n\times\frac{a+1}{a+b+1}+(1-p_n)\frac{a}{a+b+1}\text{ by}$$

hypothesis $p_n=\frac{a}{a+b}$

when $p_{n+1}=\frac{a}{a+b}\cdot\frac{a+1}{a+b+1}+\frac{b}{a+b}\times\frac{a}{a+b+1}=\frac{a}{a+1}$

$\Rightarrow\ p_n=\frac{a}{a+b}$ for all $n$.

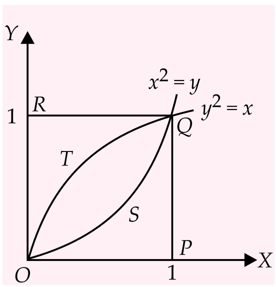
$\Rightarrow$ Choice (c), (d) are also correct.

**28. (a, b, c, d) :** $P(A) = 0.7$ ; $P(B) = 0.4$
$P(A - B) = P(A) - P(AB)\Rightarrow P(AB) = 0.2$
$\Rightarrow P(A + B) = 0.9$
$P(B - A) = 0.2$ and $P(\bar{A}\cup\bar{B})=1-P(AB)=0.8$

$$\Rightarrow\ P(B/A\cup\bar{B})=\frac{P(A\cap B)}{P(A\cup\bar{B})}=\frac{1}{4}$$

**29. (a, b, d) :** $A$ = the event of $(x, y)$ belonging to the area $OTQPO$
$B$ = the event of $(x, y)$ belonging to the area $OSQRO$

$$P(A)=\frac{\text{ar }(OTQPO)}{\text{ar }(OPQRO)}=\frac{\int_0^1\sqrt{x}dx}{1\times1}=\left[\frac{2}{3}x^{3/2}\right]_0^1=\frac{2}{3}$$



$$P(B)=\frac{\text{ar}(OSQRO)}{\text{ar}(OPQRO)}=\frac{\int_0^1\sqrt{y}dy}{1\times1}=\frac{2}{3}$$

$$P(A\cap B)=\frac{\text{ar}(OTQS)}{\text{ar}(OPQRO)}=\frac{\int_0^1\sqrt{x}\,dx-\int_0^1 x^2dx}{1\times1}=\frac{2}{3}-\frac{1}{3}=\frac{1}{3}$$

$$P(A)+P(B)=\frac{2}{3}+\frac{2}{3}\neq1.$$

So, $A$ and $B$ are not exhaustive.

$$P(A)\cdot P(B)=\frac{2}{3}\cdot\frac{2}{3}=\frac{4}{9}\neq P(A\cap B).$$

So, $A$ and $B$ are not independent.

$$P(A\cup B)=1, P(A)+P(B)=\frac{2}{3}+\frac{2}{3}\neq P(A\cup B).$$

So $A$ and $B$ are not mutually exclusive.

**30.(a, c) :** We have, $P(X=x)\propto(x+1)\left(\frac{1}{5}\right)^x$

Since, $\sum_{x=0}^{\infty}P(X=x)=1$

$$\Rightarrow\quad k\left\{1+2\left(\frac{1}{5}\right)+3\left(\frac{1}{5}\right)^2+4\left(\frac{1}{5}\right)^3+\ldots\infty\right\}=1$$

$$\Rightarrow\quad k=\frac{16}{25}$$

(a) $P(X=0)=k(0+1)\left(\frac{1}{5}\right)^0=k=\frac{16}{25}$

(b) $P(X\le1)=P(X=0)+P(X=1)$

(c) $P(X\ge1)=1-P(X=0)=1-k$

**31. (b, c) :** Here, $P(M)=m$, $P(P)=p$, $P(C)=c$

The probability of passing in at least one subject

$=1-P(\bar{M}\bar{P}\bar{C})$

$$\therefore\ \frac{75}{100}=1-(1-m)(1-p)(1-c)$$

$$\Rightarrow\ \frac{3}{4}=m+p+c-mp-pc-mc+mpc \quad \ldots(1)$$

The probability of passing in at least two subjects

$=P(MPC)+P(MP\bar{C})+P(M\bar{P}C)+P(\bar{M}PC)$

$$\therefore\ \frac{1}{2}=mpc+mp(1-c)+m(1-p)c+(1-m)pc$$

$$\Rightarrow 2mpc=mp+mc+pc-\frac{1}{2} \quad \ldots(2)$$

The probability of passing in exactly two subjects

$=P(MP\bar{C})+P(M\bar{P}C)+P(\bar{M}PC)$

$$\therefore\ \frac{2}{5}=mp(1-c)+m(1-p)c+(1-m)pc$$

$$\Rightarrow\ \frac{2}{5}=mp+mc+pc-3mpc \quad \ldots(3)$$

From (2) and (3), we get

$$2mpc+\frac{1}{2}=\frac{2}{5}+3mpc\ \Rightarrow\ mpc=\frac{1}{2}-\frac{2}{5}=\frac{1}{10}$$

From (2), we have

$$\frac{1}{5}=mp+mc+pc-\frac{1}{2}\Rightarrow mp+mc+pc=\frac{1}{5}+\frac{1}{2}=\frac{7}{10}$$

From (1), we have

$$\frac{3}{4}=m+p+c-\frac{7}{10}+\frac{1}{10}\ \Rightarrow\ m+p+c=\frac{27}{20}$$

**32. (a, b, c) :** Let the no. of blue marbles is $a$ and no. of green marbles is $b$.

$$\therefore\ \frac{ab}{^{a+b}C_2}=\frac{1}{2}\Rightarrow(a+b)(a+b-1)=4ab$$

$\Rightarrow\ b^2-(2a+1)b+a^2-a=0$

but $b\in R\Rightarrow D=(2a+1)^2-4(a^2-a)=8a+1$

$\therefore\ 8a+1$ must be a perfect square.

Hence possible values of $a$ are 3,6, 10

**33. (a, b, c, d) :** We have, $P(A\cap B)=P(A\cup B)$

$\Leftrightarrow\ P(A\cap B)=P(A)+P(B)-P(A\cap B)$

$\Leftrightarrow\ P(A)+P(B)=2P(A\cap B)$ ...(1)

We know that

$P(A\cap B)\le P(A)$ and $P(A\cap B)\le P(B)$

$2P(A\cap B)<P(A)+P(A\cap B)$

$\Rightarrow\ 2P(A\cap B)<P(A)+P(B)$ $[\because P(A\cap B)\le P(B)]$

This contradicts (1). Therefore, $P(A\cap B)=P(A)$

Similarly, $P(A\cap B)=P(B)$

Thus, $P(A)=P(B)=P(A\cap B)=P(A\cup B)$

**34. (a, b, c) :** Let $A$ be the event of getting exactly 3 defectives in the examination of 8 record players and $B$ be the event that 9th record player is defective.

$$P(A\cap B)=P(A)P\left(\frac{B}{A}\right)$$

$$P(A)=\frac{^4C_3\times{}^{11}C_5}{^{15}C_8}, P\left(\frac{B}{A}\right)=\frac{1}{7}$$

Probability of 9th one examined is the last defective

$$=\frac{^4C_3\times{}^{11}C_5}{^{15}C_8}\times\frac{1}{7}=\frac{8}{195}$$

**35. (a, b, d) :** (a) Required probability

$$=\frac{12}{12}\times\frac{11}{12}\times\frac{10}{12}\times\ldots\times\frac{1}{12}=\frac{12!}{(12)^{12}}$$

(b) Two months can be selected in ${}^{12}C_2$ ways. For each selection every person has two choices in $2^6$ ways but it includes two cases in which all persons were born in the same month.
Total number of favourable cases = ${}^{12}C_2(2^6 - 2)$

Required probability $= \dfrac{{}^{12}C_2(2^6-2)}{(12)^6}$

(d) Required probability

$= \dfrac{365}{365}\times\dfrac{364}{365}\times.....\times\dfrac{365-(n-1)}{365} = \dfrac{{}^{365}P_n}{(365)^n}$

**36. (a, b, c, d) :** $P(\bar{A}\cup\bar{B}) = P(\overline{A\cap B}) = 1 - P(A\cap B)$

but $P(\bar{A}\cup\bar{B}) = P(\bar{A}) + P(\bar{B}) - P(\bar{A}\cap\bar{B})$

$P(\bar{A}) + P(\bar{B}) - \{1 - P(A\cup B)\}$

**37. (a, b, c, d) :** $P(A \cup B) \geq P(A)$, $P(A \cup B) \geq P(B)$

$P(A\cup B)\leq 1 \Rightarrow \dfrac{3}{5}+\dfrac{2}{3}-1\leq P(A\cap B)\Rightarrow P(A\cap B)\geq\dfrac{4}{15}$

But, $P(A \cap B) \leq P(A)$ and $P(A \cap B) \leq P(B)$

$P(A\cap\bar{B}) = P(A) - P(A\cap B) \leq \dfrac{3}{5}-\dfrac{4}{15}\leq\dfrac{1}{3}$

**38. (a, b) :** There are $2^n$ out comes in all. The sequence of flips begins with $m$ successive tails followed by a head followed head or tail.

$\underbrace{T\,T\,T\,T\,T\,T\,T\,T..........\,T}_{m \text{ tails}} \qquad \underbrace{H\,x\,x\,x.....x}_{n-(m+1) \text{ tails or heads}}$

$\therefore$ Probability $= \dfrac{2^{n-(m+1)}}{2^n} = \dfrac{1}{2^{m+1}} = \dfrac{1}{2^6} \Rightarrow m = 5, n \in N$

**39. (a, b, c) :** Let $P(E) = x$, $P(F) = y$. Given

$xy = \dfrac{1}{12}$ and $(1-x)(1-y) = \dfrac{1}{2} \;\Rightarrow\; x + y = \dfrac{7}{12}$

$\Rightarrow\; x = \dfrac{1}{3}, y = \dfrac{1}{4}$ or $\;\Rightarrow\; x = \dfrac{1}{4}, y = \dfrac{1}{3}$

**40. (a, b, c, d) :** $P(A\cap B) = \dfrac{3}{6}\cdot\dfrac{3}{6} = \dfrac{1}{4}$

$P(B\cap C) = P(a \text{ is even and } b \text{ is odd}) = \dfrac{1}{4}$

$P(A\cap C) = P(a \text{ is odd and } b \text{ is even}) = \dfrac{1}{4}$

$P(A\cap B\cap C) = P(\phi) = 0$

**41. (a, c) :** $P(0\,H$ or $2\,H$ or $4\,H)$

$Q = p^4 + 6p^2(1-p)^2 - (1-p)^4 = 8p^4 - 16p^3 + 12p^2 - 4p + 1$

$= \dfrac{(2p-1)^4+1}{2}$

**42. (a, b, c) :** $1 \geq P(A) + P(B) - P(A \cap B)$

or $\quad P(A \cap B) \geq P(A) + P(B) - 1$

Let $P(A) > P(A/B)$ or $P(A) > \dfrac{P(A\cap B)}{P(B)}$

$P(A)\cdot P(B) > P(A \cap B)$

**43. (c) :** $P(E_i) = Ki(i + 1)$

$\because \quad P(E_1) + P(E_2) + .... + P(E_n) = 1$

$\Rightarrow \quad K\sum_{i=1}^{n} i(i+1) = 1$

$\Rightarrow \quad K\left[\dfrac{1}{6}n(n+1)(2n+1)+\dfrac{n(n+1)}{2}\right] = 1$

$\therefore \quad K = \dfrac{3}{n(n+1)(n+2)}$

**44. (b) :** $P(E) = \sum_{i=1}^{n} P(E_i)\cdot P(E/E_i)$

$= K\sum_{i=1}^{n} i(i+1)\cdot\dfrac{i}{n} = \dfrac{K}{n}\sum_{i=1}^{n}(i^3+i^2)$

$= \dfrac{K}{n}\left[\left(\dfrac{n(n+1)}{2}\right)^2 + \dfrac{1}{6}n(n+1)(2n+1)\right] = \dfrac{(3n+1)(n+2)}{4n(n+2)}$

$\therefore \lim\limits_{n\to\infty} P = \dfrac{3}{4}$

**45. (d) :** $P(E_1/E) = \dfrac{P(E_1)P(E/E_1)}{P(E)} = \dfrac{K\times 2\times\frac{1}{n}}{\frac{(3n+1)(n+2)}{4n(n+2)}}$

$= \dfrac{24}{n(n+1)(n+2)(3n+1)}$

**(46 - 48) :**

**46. (c)** **47. (b)** **48. (a)**

Let $E_1$ = Event that $A$ wrote a plus sign.
$E_2$ = Event that $A$ wrote a minus sign.
$E$ = Event that the referee observes a plus sign.

Given, $P(E_1) = \dfrac{1}{3} \;\Rightarrow\; P(E_2) = \dfrac{2}{3}$

$P(E / E_1)$ = Probability that none of $B$, $C$, $D$ change sign + Probability that exactly two of $B$, $C$, $D$ change sign.

$= \dfrac{1}{27} + 3\left(\dfrac{1}{3}\times\dfrac{2}{3}\times\dfrac{2}{3}\right) = \dfrac{13}{27}$

$P(E / E_2)$ = Probability that all of $B$, $C$, $D$ change the sign + Probability that exactly one of them changes the sign.

$= \dfrac{8}{27} + 3\times\left(\dfrac{2}{3}\times\dfrac{1}{3}\times\dfrac{1}{3}\right) = \dfrac{14}{27}$

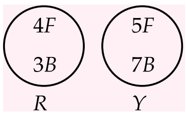
$\therefore \; P(E_1/E) = \frac{13}{41}$ [Using Baye's theorem].

**49. (c) :** The probability that the duel ends after 1st round = 1 – 0.6 × 0.8 = 0.52.

**50. (d) :** Required probability
$= 0.4 \times 0.8 + 0.6 \times 0.8 \times 0.4 \times 0.8 + 0.6 \times 0.8 \times 0.6 \times 0.8 + 0.4 \times 0.8 + \ldots$

**51. (b) :** Required probability
$= 0.4 \times 0.2 + 0.6 \times 0.8 \times 0.4 \times 0.2 + 0.6 \times 0.8 \times 0.6 \times 0.8 \times 0.4 \times 0.2 + \ldots$

$$= \frac{0.08}{1-0.48} = \frac{.08}{0.52} = \frac{8}{52} = \frac{2}{13}$$




**52. (A) → (q); (B) → (p); (C) → (s); (D) → (r)**

$$P(P \cap Q = \phi) = \left(\frac{3}{4}\right)^n$$

$$P(P \cap Q \text{ has one element}) = \frac{n \times 3^{n-1}}{4^n}$$

$$P(P \cap Q \text{ has two element}) = {}^nC_2 \frac{3^{n-2}}{4^n}$$

$$P(|P| = |Q|) = \frac{({}^nC_0)^2 + ({}^nC_1)^2 + \ldots + ({}^nC_n)^2}{4^n} = \frac{{}^{2n}C_n}{4^n}$$

**53. (A) → (r); (B) → (q); (C) → (p); (D) → (s)**

$$P(A) = \frac{18}{36},\; P(B) = 1 - \left(\frac{5}{6}\right)^2 = \frac{11}{36}$$

Now $P(A \cap B) = \frac{6}{36}$

$$P(A \cup B) = \frac{18}{36} + \frac{11}{36} - \frac{6}{36} = \frac{23}{36}$$

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

**54. (4) :** Let $n(B) = x$, $n(A \cap B) = y$

$$\Rightarrow \frac{8}{12} \cdot \frac{x}{12} = \frac{y}{12} \Rightarrow 8x = 12y$$

$\Rightarrow x = 3, 6, 9$ or $12$

**55. (5) :** Required probability $= \frac{{}^{12}C_4 \times \underline{|4}}{12^4} = \frac{55}{96}$

**56. (7) :** Total number of function $= 5^3 = 125$

Number of function satisfying $f(i) \le f(j)$ for $i < j$
$= {}^5C_3 + {}^5C_2\,(1 + 1) + {}^5C_1 = 35$

Required probability $= \frac{35}{125} = \frac{7}{25}$

**57. (8)**

**58. (2) :** $p^2 \ge 4q$

| $p$ | $q$ |
|---|---|
| 2 | 1 |
| 3 | 1, 2 |
| 4 | 1 to 4 |
| 5 | 1 to 6 |
| 6 | 1 to 9 |
| 7, 8, 9, 10 | 1 to 10 |

The total number of pairs $(p, q)$ is
1 + 2 + 4 + 6 + 9 + 40 = 62

$\therefore$ Required probability $= \frac{62}{10 \cdot 10} = \frac{31}{50}$

**59. (7) :** The sequence of consecutive heads may starts with 1st toss or 2nd toss or 3rd toss ..... In any case, if it starts with $r^{th}$ throw, the first $(r - 2)$ throws may be head or tail but $(r - 1)^{st}$ throw must be tail, after which again a head or tail can show up:

$\underbrace{\text{HHH------H}}_{7 \text{ times}} \times \times \times \times \times$

or $\underbrace{\times \times \times \text{-----} \times}_{(r-2)} \underset{(r-1)^{st}}{\underset{\downarrow}{T}}\; \underbrace{\text{HHH ------ H}} \times \times \text{-----}$

$\therefore$ Required probability =

$$\frac{1}{2^7} + \underbrace{\frac{1}{2} \cdot \frac{1}{2^7} + \frac{1}{2} \cdot \frac{1}{2^7} + \ldots + \frac{1}{2} \cdot \frac{1}{2^7}}_{5\text{-times}}$$

$$= \frac{1}{2^7}\left[1 + \frac{5}{2}\right] = \frac{7}{2^8}$$

**60. (9) :** Let $E_1$ = The toss result is a head
$E_2$ = The toss result is a tail
$A$ = Number obtained is 7 or 8

$\therefore \; P(A) = P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2)$

$$= \frac{1}{2} \times \frac{11}{36} + \frac{1}{2} \times \frac{2}{11} = \frac{193}{792}$$

$$\therefore \frac{P}{792} = \frac{193}{792}$$

$\Rightarrow P = 193$

CLASS XII Series 8

# ACE YOUR WAY CBSE

## Three Dimensional Geometry

### HIGHLIGHTS

### DIRECTION COSINES AND DIRECTION RATIOS OF A LINE

- The cosines of the angles made by a line with the positive direction of $X$, $Y$ and $Z$ axes are known as its direction cosines.
- The real numbers which are proportional to the direction cosines of a line are known as its direction ratios.
- Thus if $l, m, n$ be the direction cosines of a line and $a, b, c$ be its direction ratios, then

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \lambda, \text{ a constant}$$

**Relation between Direction cosines of a line**

If $l$, $m$ and $n$ are direction cosines of a line, then $l^2 + m^2 + n^2 = 1$

**Relation between Direction cosines and Direction ratios**

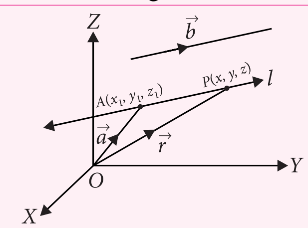
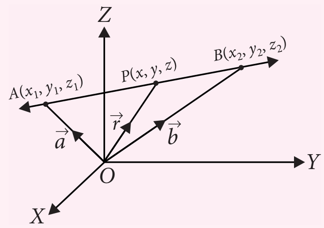
| Previous Years Analysis | | | | | | |
|---|---|---|---|---|---|---|
| | 2016 | | 2015 | | 2014 | |
| | Delhi | AI | Delhi | AI | Delhi | AI |
| VSA | 1 | – | 1 | 1 | 1 | 1 |
| SA | 1 | – | 1 | 1 | 1 | 1 |
| LA | 1 | – | 1 | 1 | 1 | 1 |

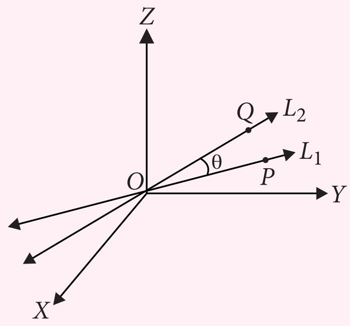
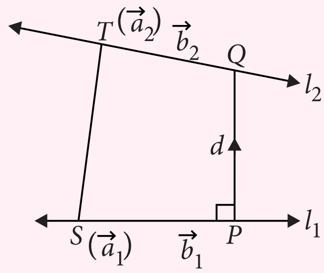
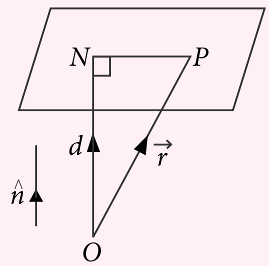
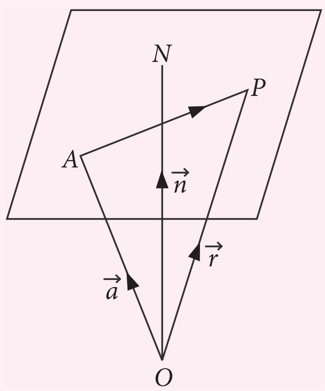
- Let $a, b, c$ be the direction ratios of a line whose direction cosines are $l, m, n$, then

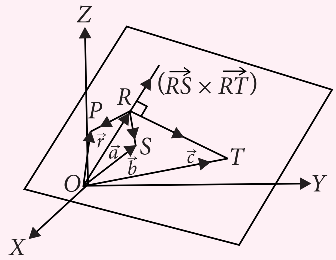
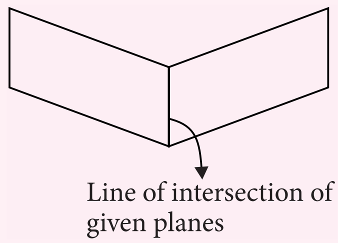
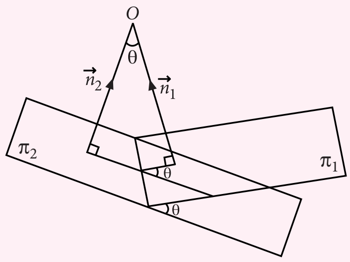
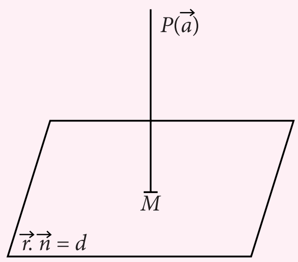
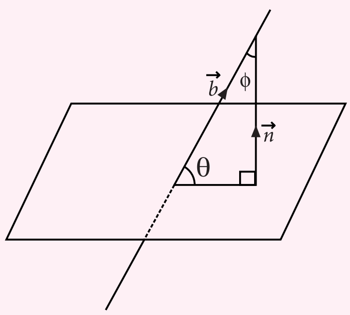
$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} \therefore\ l = \frac{\pm a}{\sqrt{a^2+b^2+c^2}},\ m = \frac{\pm b}{\sqrt{a^2+b^2+c^2}}$$

and $n = \dfrac{\pm c}{\sqrt{a^2+b^2+c^2}}$.

**Remark :** Direction ratios of the line passing through the two points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ are given by $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

| Equation | Vector Form | Cartesian form | Figure |
|---|---|---|---|
| Equation of a line passing through a given point and parallel to a given vector $\vec{b}$. | $\vec{r} = \vec{a} + \lambda\vec{b}$ | $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ If the direction cosines <$l, m, n$> are given, then equation of the line is $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ |   |
| Equation of a line passing through two given points | $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}),\ \lambda \in R$ | $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$ |   |

| | | | |
|---|---|---|---|
| Angle between two lines | Let the vector equations of two lines $L_1$ and $L_2$ be $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$<br>Then, $\cos\theta=\left\lvert\dfrac{\vec{b}_1\cdot\vec{b}_2}{\lvert\vec{b}_1\rvert\lvert\vec{b}_2\rvert}\right\rvert$ | Let the cartesian equations of two lines be $\dfrac{x-x_1}{a_1}=\dfrac{y-y_1}{b_1}=\dfrac{z-z_1}{c_1}$ and $\dfrac{x-x_2}{a_2}=\dfrac{y-y_2}{b_2}=\dfrac{z-z_2}{c_2}$<br>Then,<br>$\cos\theta=\left\lvert\dfrac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right\rvert$ |   |
| Shortest distance between two lines<br>• Distance between two skew lines<br>• Distance between two parallel lines | Let $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$ represents two skew lines, then the magnitude of the shortest distance ($d$) is<br>$\left\lvert\dfrac{(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2-\vec{a}_1)}{\lvert\vec{b}_1\times\vec{b}_2\rvert}\right\rvert$<br>$\left\lvert\dfrac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{\lvert\vec{b}\rvert}\right\rvert$ | The shortest distance ($d$) between the lines, $\dfrac{x-x_1}{a_1}=\dfrac{y-y_1}{b_1}=\dfrac{z-z_1}{c_1}$ and $\dfrac{x-x_2}{a_2}=\dfrac{y-y_2}{b_2}=\dfrac{z-z_2}{c_2}$ is<br>$d=\left\lvert\dfrac{\begin{vmatrix}x_2-x_1 & y_2-y_1 & z_2-z_1\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}\right\rvert$ |   |
| • Condition for Coplanarity of two lines | $(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)=0$ | $\begin{vmatrix}x_2-x_1 & y_2-y_1 & z_2-z_1\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}=0$ | |
| Equation of a plane in<br>• Normal form<br>• Intercept form | $\vec{r}\cdot\hat{n}=d$ | $lx+my+nz=d$<br>$\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$ |   |
| Equation of a plane perpendicular to a given vector and passing through a given point | Let $\vec{r}$ be the position vector of any point $P(x, y, z)$ in the plane, then $(\vec{r}-\vec{a})\cdot\vec{n}=0$ | $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ |   |

| | | | |
|---|---|---|---|
| Equation of a plane passing through three non-collinear points | $(\vec{r}-\vec{a})\cdot[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})]=0$ | $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$ |  |
| Equation of the plane passing through the intersection of two given planes | $\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2)=d_1+\lambda d_2$ | $(a_1x+b_1y+c_1z-d_1)+\lambda(a_2x+b_2y+c_2z-d_2)=0$ |  |
| Angle between two planes | $\cos\theta=\left\|\dfrac{\vec{n}_1\cdot\vec{n}_2}{\|\vec{n}_1\|\|\vec{n}_2\|}\right\|$ | $\cos\theta=\left\|\dfrac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right\|$<br>planes to be<br>(i) perpendicular, if<br>$\cos\theta = a_1a_2+b_1b_2+c_1c_2=0$<br>(ii) parallel, if $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$ |  |
| Distance of a point from a plane | $\dfrac{\|\vec{a}\cdot\vec{n}-d\|}{\|\vec{n}\|}$, where $\vec{n}$ is normal to the plane. | $\left\|\dfrac{ax_1+by_1+cz_1-d}{\sqrt{a^2+b^2+c^2}}\right\|$ |  |
| Angle between a line and a plane | $\theta=\sin^{-1}\left\|\dfrac{\vec{b}\cdot\vec{n}}{\|\vec{b}\|\|\vec{n}\|}\right\|$ | $\sin\theta=\left\|\dfrac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}\right\|$ |  |

## PROBLEMS

### Very Short Answer Type

1. Find the acute angle between the lines whose direction ratios are (2, 3, 6) and (1, 2, 3).

2. If $\alpha, \beta, \gamma$ are the direction angles of a vector and $\cos\alpha = \frac{14}{15}$, $\cos\beta = \frac{1}{3}$, then find $\cos\gamma$.

3. Show that the planes $2x + 6y + 6z = 7$ and $3x + 4y - 5z = 8$ are at right angles.

4. A line passes through the points (3, 1, 2) and (5, –1, 1). Find the direction ratios of the line.

5. Reduce the equation of the plane $2x + 3y - 4z = 12$ to intercept form and find its intercepts on the coordinate axes.

### Short Answer Type

6. Find the cartesian equation of the line passing through the point (2, 3, –4) and perpendicular to $XZ$-plane.

7. Find a unit vector normal to the plane $\vec{r}\cdot(2\hat{i} - 3\hat{j} + 6\hat{k}) + 14 = 0$

8. Find the cartesian equation of the plane passing through the points (1, 1, 2), (0, 2, 3), (4, 5, 6).

9. Find the vector equation of the line passing through the points $A(3, 4, -7)$ and $B(6, -1, 1)$.

10. Find the shortest distance between the lines
$\vec{r} = (\hat{i} + 2\hat{j}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$ and
$\vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 6\hat{k})$

### Long Answer Type - I

11. Show that the lines $\frac{x-2}{1} = \frac{y-4}{4} = \frac{z-6}{7}$ and $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ are coplanar.

12. Find the direction ratios of the normal to the plane passing through the point (2, 1, 3) and the line of intersection of the planes $x + 2y + z = 3$ and $2x - y - z = 5$.

13. If $l_1, m_1, n_1$ and $l_2, m_2, n_2$ be the direction cosines of two lines, show that the direction cosines of the line perpendicular to both of them are proportional to $(m_1n_2 - m_2n_1), (n_1l_2 - n_2l_1), (l_1m_2 - l_2m_1)$.

14. Find the length of perpendicular from origin to the plane $2x - 3y + 6z - 14 = 0$.

15. If the points $(1, 1, \lambda)$ and $(-3, 0, 1)$ be equidistant from the plane $\vec{r}\cdot(3\hat{i} + 4\hat{j} - 12\hat{k}) + 13 = 0$, find the value of $\lambda$.

### Long Answer Type - II

16. If the perpendicular distance of a plane from the origin is 1 and d.c.s of normal vector to the plane satisfies $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = k$, then find the value of $k$.

17. If this lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then find the value of $k$ and hence find the equation of the plane containing these lines.

18. Show that the angles between the diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$.

19. Consider the lines

$$L_1 : \frac{x+1}{3} = \frac{y+2}{1} = \frac{z+1}{2}$$

$$L_2 : \frac{x-2}{1} = \frac{y+2}{2} = \frac{z-3}{3}$$

Find the distance of the point (1, 1, 1) from the plane passing through the point (– 1, – 2, –1) and whose normal is perpendicular to both the lines $L_1$ and $L_2$.

20. Find the equations of two planes through the points (4, 2, 1), (2, 1, – 1) and making angles $\frac{\pi}{4}$ with the plane $x - 4y + z - 9 = 0$.

## SOLUTIONS

1. Here, $a_1 = 2, b_1 = 3, c_1 = 6\,;\, a_2 = 1, b_2 = 2, c_2 = 3$

$$\therefore\ \cos\theta = \left|\frac{(2)(1)+(3)(2)+(6)(3)}{\sqrt{2^2+3^2+6^2}\sqrt{1^2+2^2+3^2}}\right| = \frac{26}{7\sqrt{14}}$$

$$\Rightarrow\ \theta = \cos^{-1}\left(\frac{26}{7\sqrt{14}}\right)$$

2. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$\Rightarrow\ \frac{196}{225} + \frac{1}{9} + \cos^2\gamma = 1$$

$\Rightarrow \cos^2\gamma = 1 - \frac{221}{225} = \frac{4}{225} \Rightarrow \cos\gamma = \pm\frac{2}{15}$

**3.** We have, planes $2x + 6y + 6z = 7$ and $3x + 4y - 5z = 8$
Here, $a_1 = 2, b_1 = 6, c_1 = 6, a_2 = 3, b_2 = 4, c_2 = -5$
$\therefore\ a_1a_2 + b_1b_2 + c_1c_2 = (2)(3) + (6)(4) + (6)(-5) = 0$
So, planes are at right angles.

**4.** Direction ratios of the line are
$< 5 - 3, -1 -1, 1 - 2 >$ or $< 2, -2, -1 >$

**5.** The equation of the given plane is
$2x + 3y - 4z = 12 \Rightarrow \frac{2x}{12} + \frac{3y}{12} - \frac{4z}{12} = 1$

$\Rightarrow \frac{x}{6} + \frac{y}{4} + \frac{z}{-3} = 1$

So, the intercepts made by the plane with the coordinate axes are 6, 4 and $-3$ respectively.

**6.** Let $A \equiv (2, 3, -4)$.
Then, the position vector $\vec{a}$ of $A$ is $\vec{a} = 2\hat{i} + 3\hat{j} - 4\hat{k}$
Since, the line is perpendicular to $XZ$-plane, therefore it is parallel to $Y$-axis.
$\therefore$ The line is parallel to the unit vector $\hat{j}$.
$\therefore$ Vector equation of the required line is
$\vec{r} = (2\hat{i} + 3\hat{j} - 4\hat{k}) + \lambda\hat{j}$

$\Rightarrow x\hat{i} + y\hat{j} + z\hat{k} = (2\hat{i} + 3\hat{j} - 4\hat{k}) + \lambda\hat{j}$

$\Rightarrow \frac{x-2}{0} = \frac{y-3}{1} = \frac{z+4}{0}$

**7.** The equation of the given plane is
$$\vec{r}\cdot(2\hat{i} - 3\hat{j} + 6\hat{k}) + 14 = 0$$
$\Rightarrow \vec{r}\cdot(2\hat{i} - 3\hat{j} + 6\hat{k}) = -14$

$\Rightarrow \vec{r}\cdot(-2\hat{i} + 3\hat{j} - 6\hat{k}) = 14$

$\Rightarrow \vec{r}\cdot\vec{n} = 14$, where $\vec{n} = (-2\hat{i} + 3\hat{j} - 6\hat{k})$

$|\vec{n}| = \sqrt{(-2)^2 + 3^2 + (-6)^2} = 7$

Hence, the unit vector normal to the given plane is
$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{-2\hat{i} + 3\hat{j} - 6\hat{k}}{7} = \left(-\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}\right)$$

**8.** Let $A(x_1, y_1, z_1) \equiv (1, 1, 2)$,
$B(x_2, y_2, z_2) \equiv (0, 2, 3)$, $C(x_3, y_3, z_3) \equiv (4, 5, 6)$
Equation of plane is
$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z-2 \\ -1 & 1 & 1 \\ 3 & 4 & 4 \end{vmatrix} = 0$$
$\Rightarrow (x - 1)(4 - 4) - (y - 1)(-4 - 3) + (z - 2)(-4 - 3) = 0$
$\Rightarrow y - 1 - z + 2 = 0 \Rightarrow y - z + 1 = 0$

**9.** Here $A \equiv (3, 4, -7)$, $B \equiv (6, -1, 1)$, then
$\vec{a} = 3\hat{i} + 4\hat{j} - 7\hat{k},\ \vec{b} = 6\hat{i} - \hat{j} + \hat{k}$
Now $\vec{b} - \vec{a} = 3\hat{i} - 5\hat{j} + 8\hat{k}$
Required vector equation is, $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$
$\Rightarrow \vec{r} = (3\hat{i} + 4\hat{j} - 7\hat{k}) + \lambda(3\hat{i} - 5\hat{j} + 8\hat{k})$

**10.** Here $\vec{a_1} = \hat{i} + 2\hat{j},\ \vec{a_2} = 3\hat{i} + 3\hat{j} - 5\hat{k},\ \vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$
Now, $(\vec{a_2} - \vec{a_1}) = 2\hat{i} + \hat{j} - 5\hat{k}$
$\therefore\ \vec{b} \times (\vec{a}_2 - \vec{a}_1) = -21\hat{i} + 22\hat{j} - 4\hat{k}$
$\therefore\ |\vec{b} \times (\vec{a}_2 - \vec{a}_1)| = \sqrt{441 + 484 + 16} = \sqrt{941}$
and $|\vec{b}| = \sqrt{(2)^2 + (3)^2 + (6)^2} = \sqrt{49} = 7$
$\therefore\ d = \frac{\sqrt{941}}{7}$ units.

**11.** The given lines are
$\frac{x-2}{1} = \frac{y-4}{4} = \frac{z-6}{7}$ and $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$
Here, $x_1 = 2, y_1 = 4, z_1 = 6, x_2 = -1, y_2 = -3, z_2 = -5,$
$a_1 = 1, b_1 = 4, c_1 = 7, a_2 = 3, b_2 = 5, c_2 = 7$
$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} -3 & -7 & -11 \\ 1 & 4 & 7 \\ 3 & 5 & 7 \end{vmatrix}$$
$= (-3)(28 - 35) + 7(7 - 21) - 11(5 - 12)$
$= 21 - 98 + 77 = 0$
$\therefore$ Given lines are coplanar.

**12.** The equation of the plane passing through the line of intersection of the planes $x + 2y + z = 3$ and $2x - y - z = 5$ is given by
$(x + 2y + z - 3) + \lambda(2x - y - z - 5) = 0$
$\Rightarrow x(2\lambda + 1) + y(2 - \lambda) + z(1 - \lambda) - 3 - 5\lambda = 0$ ...(i)
It passes through $(2, 1, 3)$, then
$2(2\lambda + 1) + (2 - \lambda) + 3(1 - \lambda) - 3 - 5\lambda = 0$
$\Rightarrow 4\lambda + 2 + 2 - \lambda + 3 - 3\lambda - 3 - 5\lambda = 0$
$\Rightarrow 4 - 5\lambda = 0 \Rightarrow \lambda = \frac{4}{5}$
Substituting $\lambda = \frac{4}{5}$ in (i), we get $13x + 6y + z - 35 = 0$ as the equation of the required plane. Clearly, direction

ratios of normal to this plane are proportional to 13, 6, 1.

**13.** Let $l$, $m$, $n$ be the direction cosines of the line perpendicular to each of the given lines. Then,

$ll_1 + mm_1 + nn_1 = 0$ ...(i)

and $ll_2 + mm_2 + nn_2 = 0$ ...(ii)

On solving (i) and (ii) we get

$$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$$

Hence, the direction cosines of the line perpendicular to the given lines are proportional to $(m_1n_2 - m_2n_1)$, $(n_1l_2 - n_2l_1)$, $(l_1m_2 - l_2m_1)$.

**14.** Given, equation of plane is $2x - 3y + 6z - 14 = 0$

Equation of plane in normal form is

$$\frac{2x}{\sqrt{2^2+(-3)^2+6^2}} + \frac{(-3)y}{\sqrt{2^2+(-3)^2+6^2}} + \frac{6z}{\sqrt{2^2+(-3)^2+6^2}}$$
$$= \frac{14}{\sqrt{2^2+(-3)^2+(6)^2}}$$

$$\Rightarrow \frac{2}{7}x - \frac{3}{7}y + \frac{6}{7}z = 2$$

So, length of perpendicular from origin to the plane is 2 units

**15.** It is given that the points $(1, 1, \lambda)$ and $(-3, 0, 1)$ are equidistant from the plane $\vec{r}\cdot(3\hat{i}+4\hat{j}-12\hat{k})+13=0$.

$$\therefore \left|\frac{(\hat{i}+\hat{j}+\lambda\hat{k})\cdot(3\hat{i}+4\hat{j}-12\hat{k})+13}{\sqrt{9+16+144}}\right| = \left|\frac{(-3\hat{i}+0\hat{j}+\hat{k})\cdot(3\hat{i}+4\hat{j}-12\hat{k})+13}{\sqrt{9+16+144}}\right|$$

$$\Rightarrow \left|\frac{3+4-12\lambda+13}{13}\right| = \left|\frac{-9+0-12+13}{13}\right|$$

$\Rightarrow |20 - 12\lambda| = 8 \quad \Rightarrow 20 - 12\lambda = \pm 8$

$\Rightarrow 20 - 12\lambda = 8$ or, $20 - 12\lambda = -8$

$\Rightarrow 12\lambda = 12$ or, $12\lambda = 28 \Rightarrow \lambda = 1$ or, $\lambda = \frac{7}{3}$

**16.** Let d.c. of the normal to the plane be $< l, m, n >$, then its equation can be written as

$lx + my + nz = 1$ ...(i)

or $lx + my + nz - 1 = 0$

Also, $l^2 + m^2 + n^2 = 1$ ...(ii)

It is given that $< l, m, n >$ satisfy the equation

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} = k \Rightarrow \frac{1}{l^2}+\frac{1}{m^2}+\frac{1}{n^2} = k$$

$$\Rightarrow (l^2+m^2+n^2)\left\{\frac{1}{l^2}+\frac{1}{m^2}+\frac{1}{n^2}\right\} = k$$

As A.M. $\geq$ H. M., therefore,

$$\frac{l^2+m^2+n^2}{3} \geq \frac{3}{\frac{1}{l^2}+\frac{1}{m^2}+\frac{1}{n^2}}$$

$$\Rightarrow (l^2+m^2+n^2)\left(\frac{1}{l^2}+\frac{1}{m^2}+\frac{1}{n^2}\right) \geq 9$$

$\Rightarrow k \geq 9$.

**17.** Let $x_1 = 1$, $y_1 = -1$, $z_1 = 1$, $x_2 = 3$, $y_2 = k$, $z_2 = 0$, $l_1 = 2$, $m_1 = 3$, $n_1 = 4$, $l_2 = 1$, $m_2 = 2$, $n_2 = 1$

If given lines intersect, then they must be coplanar.

$$\therefore \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 3-1 & k+1 & 0-1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow 2(3 - 8) - (k + 1)(2 - 4) - 1(4 - 3) = 0$

$\Rightarrow -10 + 2k + 2 - 1 = 0 \Rightarrow 2k - 9 = 0 \Rightarrow k = \frac{9}{2}$

The equation of the plane containing the given lines is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y+1 & z-1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

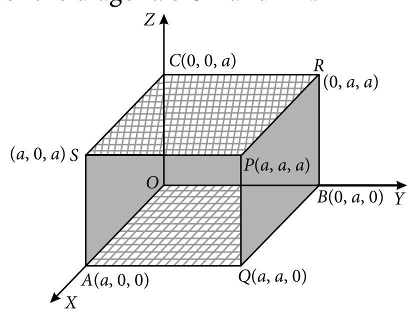
$\Rightarrow (x - 1)(3 - 8) - (y + 1)(2 - 4) + (z - 1)(4 - 3) = 0$

$\Rightarrow -5x + 5 + 2y + 2 + z - 1 = 0$

$\Rightarrow 5x - 2y - z = 6$

**18.** Let $a$ be the length of an edge of the cube and let one corner be at the origin as shown in figure. Clearly, $OP$, $AR$, $BS$ and $CQ$ are the diagonals of the cube.

Consider the diagonals $OP$ and $AR$.



Direction ratios of $OP$ and $AR$ are proportional to $a-0, a-0, a-0$ and $0-a, a-0, a-0$ *i.e.*, $a, a, a$ and $-a, a, a$ respectively.

Let $\theta$ be the angle between $OP$ and $AR$. Then,

$$\cos\theta=\left|\frac{a\times -a+a\times a+a\times a}{\sqrt{a^2+a^2+a^2}\sqrt{(-a)^2+a^2+a^2}}\right|$$

$$\Rightarrow \cos\theta=\left|\frac{-a^2+a^2+a^2}{\sqrt{3a^2}\sqrt{3a^2}}\right| \Rightarrow \cos\theta=\frac{1}{3}$$

$$\Rightarrow \theta=\cos^{-1}\left(\frac{1}{3}\right)$$

Similarly, the angles between the other pairs of diagonals are each equal to $\cos^{-1}\left(\frac{1}{3}\right)$.

Hence, the angle between any two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$.

**19.** Any plane through $(-1, -2, -1)$ is

$A(x+1)+B(y+2)+C(z+1)=0$ ...(i)

D.r.'s of any normal to (i) are $< A, B, C >$.

As this normal is at right angle to both $L_1$ and $L_2$, therefore,

$3A+1B+2C=0$ ...(ii)

$1A+2B+3C=0$ ...(iii)

Eliminating $A$, $B$, $C$ between (i), (ii) and (iii), we obtain

$$\begin{vmatrix} x+1 & y+2 & z+1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix}=0$$

$\Rightarrow (x+1)(3-4)-(y+2)(9-2)+(z+1)(6-1)=0$

$\Rightarrow -(x+1)-7(y+2)+5(z+1)=0$

or $x+7y-5z+10=0$ ...(iv)

$\therefore$ Distance of $(1, 1, 1)$ from the plane (iv)

$$=\frac{|1+7-5+10|}{\sqrt{1^2+7^2+(-5)^2}}=\frac{13}{\sqrt{75}} \text{ units.}$$

**20.** Any plane through the point $(4, 2, 1)$ is

$A(x-4)+B(y-2)+C(z-1)=0$ ...(i)

Since it passes through the point $(2, 1, -1)$, we have

$A(2-4)+B(1-2)+C(-1-1)=0$

or $2A+B+2C=0$ ...(ii)

As plane (i) makes an angle of $\frac{\pi}{4}$ with the plane $x-4y+z-9=0$, we have

$$\frac{|1A+(-4)B+1C|}{\sqrt{1+16+1}\sqrt{A^2+B^2+C^2}}=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

or $|A-4B+C|=\frac{1}{\sqrt{2}}\sqrt{18}\sqrt{A^2+B^2+C^2}$

$=3\sqrt{A^2+B^2+C^2}$

or $9(A^2+B^2+C^2)=(A-4B+C)^2$ ...(iii)

Putting $B=-2A-2C$ from (ii) in (iii), we get

$9(A^2+4A^2+4C^2+8AC+C^2)=(9A+9C)^2$

or $5A^2+5C^2+8AC=9(A^2+C^2+2AC)$

or $2A^2+5AC+2C^2=0$

or $(2A+C)(A+2C)=0$

$\Rightarrow A=-\frac{1}{2}C$ or $-2C$

From (ii), when $A=-\frac{1}{2}C, B=-C$ and when $A=-2C, B=2C$.

Substituting these values in (i), we get

$-\frac{1}{2}C(x-4)-C(y-2)+C(z-1)=0$

$\Rightarrow x+2y-2z-6=0$

and $-2C(x-4)+2C(y-2)+C(z-1)=0 (C\neq 0)$

$\Rightarrow 2x-2y-z-3=0$

# Challenging PROBLEMS ON Vectors and 3D Geometry

**1.** If $k$ is the length of any edge of a regular tetrahedron then the distance of any vertex from the opposite face is

(a) $\sqrt{\frac{2}{3}}k$ (b) $\frac{2}{3}k$ (c) $\frac{\sqrt{3}}{2}k$ (d) $\frac{1}{\sqrt{3}}k$

**2.** Let $A$ and $B$ be opposite vertices of a cube of edge length 1 unit. The radius of the sphere with centre inside the cube, tangent to the three faces meeting at $A$ and tangent to the three edges meeting at $B$ is

(a) $\sqrt{2}-1$ (b) $\sqrt{3}-\sqrt{2}$

(c) $2-\sqrt{2}$ (d) $\sqrt{5}-\sqrt{3}$

**3.** In the tetrahedron $ABCD$, we have $AB$, $CD < 1$ unit, $AC = BD = 1$ unit and $AD$, $BC > 1$ unit. The maximum radius of the sphere inscribed in the tetrahedron is

(a) $\frac{\sqrt{3}}{8}$ (b) $\frac{\sqrt{5}}{8}$ (c) $\frac{\sqrt{7}}{8}$ (d) $\frac{\sqrt{10}}{8}$

**4.** Let $\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4$ be the vectors whose magnitudes are respectively equal to areas of faces $F_1$, $F_2$, $F_3$, $F_4$ of a tetrahedron and whose directions are perpendicular to these faces in outward direction then $|\vec{n}_1+\vec{n}_2+\vec{n}_3+\vec{n}_4| =$

(a) 0 (b) 1 (c) 2 (d) 4

**5.** A rectangular parallelopiped has sides of length $a$, $b$, $c$. The shortest distance of the edge of length '$a$' from the diagonal (not meeting it) is

(a) $\frac{a^2}{\sqrt{a^2+b^2+c^2}}$ (b) $\frac{a(b+c)}{\sqrt{a^2+b^2+c^2}}$

(c) $\frac{bc}{\sqrt{b^2+c^2}}$ (d) $\frac{a^2}{\sqrt{b^2+c^2}}$

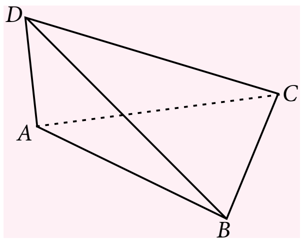
**6.** Given $\Delta ABC$ and $\Delta AEF$ such that $B$ is the midpoint of $EF$. Also, $AB = EF = 1$, $BC = 6$, $CA = \sqrt{33}$ and $\overline{AB}\cdot\overline{AE}+\overline{AC}\cdot\overline{AF} = 2$. The cosine of the angle between $\overline{BC}$ and $\overline{EF}$ is

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{\sqrt{3}}{2}$

**7.** $ABCDA'B'C'D'$ is a rectangular parallelopiped with $AA' = 2AB = 8a$. $E$ is the midpoint of $AB$ and $M$ is the point on $DD'$ such that $DM = a\left(1+\frac{AD}{AC}\right)$. Let $F$ be the point on the segment $AA'$ for which $CF + FM$ has minimum possible value then $AF =$

(a) $a$ (b) $2a$ (c) $3a$ (d) $4a$

**8.** As shown in the diagram, volume of tetrahedron $ABCD = 1/6$ and $\angle ACB = 45°$ with $AD+BC+\frac{AC}{\sqrt{2}} = 3$ then $CD =$




(a) 1 (b) $\sqrt{3}$ (c) 2 (d) $\sqrt{2}$

**9.** Let $ABCD$ be an arbitrary tetrahedron. The bisectors of the angles $\angle BDC$, $\angle CDA$, $\angle ADB$ intersect $BC$, $CA$, $AB$ respectively in points $M$, $N$, $P$. Consider planes $\pi_1 : ADM$, $\pi_2 : BDN$ and $\pi_3 : CDP$

(a) $\pi_1, \pi_2, \pi_3$ do not meet

(b) $\pi_1, \pi_2, \pi_3$ all meet on a common line

(c) $\pi_1, \pi_2$ meet on a common line but $\pi_3$ does not meet this line

(d) $\pi_1, \pi_3$ meet on a common line but $\pi_2$ does not meet this line.

**10.** Consider the points $A$, $B$, $C$, $D$ in a plane (no three of them being collinear). $H_1$ and $H_2$ are orthocentres of

By : Tapas Kr. Yogi, Mob : 9533632105.

triangles $ABC$ and $ABD$ respectively. The points $A, B, C, D$ are concyclic if

(a) $\overrightarrow{H_1H_2} = \overrightarrow{CD}$ (b) $\overrightarrow{H_1H_2} = 2\overrightarrow{CD}$

(c) $\overrightarrow{H_1H_2} = 3\overrightarrow{CD}$ (d) $\overrightarrow{H_1H_2} = \frac{2}{3}\overrightarrow{CD}$

**11.** Let $ABC$ be an equilateral triangle. The perpendiculars $AA'$ and $BB'$ on the plane containing $ABC$ at the points $A$ and $B$ are such that $AA' = AB$ and $BB' = \frac{1}{2}AB$ then find $\angle ACD$ where $D$ is the point of intersection of lines $AB$ and $A'B'$.

(a) 45° (b) 60°

(c) 90° (d) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

**12.** A rectangle $ABCD$ of dimension $r$ and $2r$ is folded along diagonal $BD$ such that planes $ABD$ and $CBD$ are perpendicular to each other. Let $C$ becomes $C'$ in new position, then the distance $AC'$ is

(a) $\frac{\sqrt{85}}{5}r$ (b) $\frac{\sqrt{35}}{5}r$ (c) $\frac{\sqrt{65}}{5}r$ (d) $\frac{\sqrt{91}}{5}r$

**13.** Points $A, B, C, D$ are taken such that they are not in the same plane and $M$ is the midpoint of $BC$ such that $AB = BD = CD = AC = \sqrt{2}AD = \frac{BC}{\sqrt{2}} = a$ then

(a) $MA = MB$ (b) $MA \neq MD$

(c) $MA = \frac{1}{2}MB$ (d) $MA = 2MC$

**14.** In tetrahedron $OABC$, let $\angle BOC = \alpha$, $\angle COA = \beta$, $\angle AOB = \gamma$ then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma - 1$ is

(a) positive (b) negative

(c) depends on $(\alpha, \beta, \gamma)$

(d) zero

**15.** Let $ABCD$ be a rhombus. $M, N, P$ are points on the sides $AB$, $BC$, $CD$ respectively. The centroid of the triangle $MNP$ lies on the line $AC$ if

(a) $AM + DP = BN$ (b) $AM + 2DP = BN$

(c) $AM + BN = DP$ (d) $AM + BN = 2DP$

**16.** Let $M, N, P, Q$ be the midpoints of the edges $AB$, $CD$, $AC$, $BD$ respectively of the tetrahedron $ABCD$. It is known that $MN$ is perpendicular to both $AB$ and $CD$ then

(a) $AC = BD$ (b) $AC = \frac{1}{2}BD$

(c) $AC = 2BD$ (d) $4AC = BD$

**17.** Let $ABC$ be a right triangle with $\angle A = 90°$. Two perpendiculars on the triangles plane are erected at points $A$ and $B$ and the points $M$ and $N$ are considered on these perpendiculars, on the same side of the plane such that $BN < AM$ and $AC = 2a$, $AB = a\sqrt{3}$, $AM = a$ and angle between planes $MNC$ and $ABC$ equals 30°. The area of the $\Delta MNC$ is

(a) $a^2$ (b) $\sqrt{2}a^2$ (c) $2a^2$ (d) $4a^2$

**18.** $O$ is the centre, $AB$ and $BC$ are two diagonals of the adjacent faces of a rectangular box. If angles $AOB$, $BOC$ and $COA$ are $\alpha, \beta, \theta$ then $\cos\alpha + \cos\beta + \cos\theta =$

(a) $-1$ (b) 0 (c) 1 (d) 3/2

**19.** Let $AB$ be a diameter of a circle and $\pi$ is a plane through $AB$ making an angle $\theta$ with the plane of circle. If diameter of the circle is $2a$ then the eccentricity of the curve of the projection of the circle on $\pi$ is

(a) $r\sin\theta$ (b) $\sin\theta$ (c) $\cos\theta$ (d) $r\cos\theta$

## SOLUTIONS

**1. (a) :** Let $OABC$ be the tetrahedron of edge length '$k$' each.

Let $M$ be the midpoint of $AB$ then $AM = k/2$

Let $N$ be the foot of the perpendicular from $C$ on the plane $OAB$ then

$$AN = \frac{k}{\sqrt{3}}. \text{ So } CN = \sqrt{k^2 - \left(\frac{k}{\sqrt{3}}\right)^2} = \sqrt{\frac{2}{3}}k.$$

**2. (c) :** Let $A(0, 0, 0)$, $B(1, 1, 1)$ and the edges be parallel to the co-ordinate axes. Let $r$ be the radius of the sphere then centre is $(r, r, r)$ and $(1, r, 1)$ is the point of tangency of one of the edges at $B$.

So, $r^2 = 2(1 - r)^2$ giving $r = 2 - \sqrt{2}$.

**3. (a) :** Let $h$ be the perpendicular length on a side of length (> 1), then equating areas of base,

$$\frac{1}{2}ah = \frac{1}{2}\cdot 1\cdot b\sin 60°.$$

Where $a > 1$, $b < 1$ then $h < \frac{\sqrt{3}}{2}$.

Let $r$ be the radius of in-sphere.

Now, volume $V = \frac{r}{3}(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4) \le \frac{r}{3}(4\Delta_1)$

*i.e.* $r > \frac{3V}{4\Delta_1}$. Also, $V = \frac{1}{3}h\cdot\Delta_1 < \frac{\sqrt{3}}{2}\cdot\frac{\Delta_1}{3} = \frac{\Delta_1}{2\sqrt{3}}$

So, $r \ge \frac{3}{4\Delta_1}\cdot\frac{\Delta_1}{2\sqrt{3}} = \frac{\sqrt{3}}{8}$

**4. (a) :** Let $O(\vec{0})$, $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$ be the vertices of the tetrahedron then,

$$\vec{n}_1=\frac{1}{2}(\vec{a}\times\vec{b}),\ \vec{n}_2=\frac{1}{2}(\vec{b}\times\vec{c}),\ \vec{n}_3=\frac{1}{2}(\vec{c}\times\vec{a})$$

and $\vec{n}_4=\frac{1}{2}(\vec{c}-\vec{a})\times(\vec{b}-\vec{a})$

Hence, $\vec{n}_1+\vec{n}_2+\vec{n}_3+\vec{n}_4=\vec{0}$.

**5. (c) :** Let the edges be along the axes and $A(a, 0, 0)$, $B(0, b, 0)$, $D(a, 0, c)$ where $O(0, 0, 0)$ then

$$\overrightarrow{OA}=a\hat{i},\ \overrightarrow{BD}=a\hat{i}-b\hat{j}+c\hat{k}$$

So, shortest distance $=\left|\dfrac{\overrightarrow{OB}\cdot(\overrightarrow{OA}\times\overrightarrow{BD})}{|\overrightarrow{OA}\times\overrightarrow{BD}|}\right|=\dfrac{bc}{\sqrt{b^2+c^2}}$

**6. (b) :** Let $\theta$ be the required angle.

Now, $\overrightarrow{AC}\cdot\overrightarrow{AB}=\sqrt{33}\times1\times\dfrac{33+1-36}{2\times1\times\sqrt{33}}=-1$

and $\overrightarrow{BE}=-\overrightarrow{BF}$ and $\overrightarrow{AB}^2=1$

So, given expression becomes

$\overrightarrow{AB}\cdot\overrightarrow{AE}+\overrightarrow{AC}\cdot\overrightarrow{AF}=2$

$=\overrightarrow{AB}\cdot(\overrightarrow{AB}+\overrightarrow{BE})+\overrightarrow{AC}\cdot(\overrightarrow{AB}+\overrightarrow{BF})$

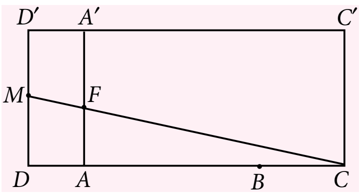
*i.e.* $\overrightarrow{AB}^2+\overrightarrow{AB}\cdot\overrightarrow{BE}+\overrightarrow{AC}\cdot\overrightarrow{AB}+\overrightarrow{AC}\cdot\overrightarrow{BF}=2$

*i.e.* $1+\overrightarrow{BF}\cdot(\overrightarrow{AC}-\overrightarrow{AB})-1=2$ or $\overrightarrow{BF}\cdot\overrightarrow{BC}=2$

*i.e.* $|\overrightarrow{BF}|\cdot|\overrightarrow{BC}|\cos\theta=2$

Hence, $\cos\theta=\dfrac{2}{3}$

**7. (a) :** Opening up the parallelopiped so that $C$ and $D$ come in line with $AB$ as shown in figure.




For $CF + FM$ to be minimum, we see that $F$ must be the common point of $AA'$ and $CM$.

So, by similar triangles,

$$AF=DM\cdot\frac{AC}{CD}=a\cdot\left(1+\frac{AD}{AC}\right)\frac{AC}{AD+AC}$$

$\Rightarrow AF = a$

**8. (b) :** $V_{ABCD}\le\dfrac{1}{3}\cdot AD\cdot\left(\dfrac{1}{2}\cdot BC\cdot AC\cdot\sin45^\circ\right)$

*i.e.*, $\dfrac{1}{6}\le\dfrac{1}{3}\cdot AD\cdot\dfrac{1}{2}\cdot BC\cdot\dfrac{AC}{\sqrt{2}}$ *i.e.*, $AD\cdot BC\cdot\dfrac{AC}{\sqrt{2}}\ge1$

Now, apply A.M. ≥ G.M. on $\left(AD,\ BC,\ \dfrac{AC}{\sqrt{2}}\right)$,

we have, $\dfrac{AD+BC+\dfrac{AC}{\sqrt{2}}}{3}\ge\sqrt[3]{AD\cdot BC\cdot\dfrac{AC}{\sqrt{2}}}$

*i.e.*, $\dfrac{3}{3}\ge\sqrt[3]{1}$ Hence, $AD=BC=\dfrac{AC}{\sqrt{2}}$

So, $AD$ is perpendicular to face $ABC$.

So, $DC=\sqrt{AD^2+AC^2}=\sqrt{3}$.

**9. (b) :** Since $DM$ is the bisector of $\angle BDC$, we have

$$\frac{BM}{CM}=\frac{BD}{CD}$$

Similarly, $\dfrac{CN}{AN}=\dfrac{CD}{AD},\ \dfrac{AP}{BP}=\dfrac{AD}{BD}$

Hence, $\dfrac{BM}{CM}\cdot\dfrac{CN}{AN}\cdot\dfrac{AP}{BP}=1$

and from Ceva's theorem, it follows that $AM$, $BN$, $CP$ have a common point $Q$.

So, $\pi_1\cap\pi_2\cap\pi_3 = DQ$ line

**10. (a) :** Let $O_1$, $O_2$ be the circumcentres of triangles $ABC$ and $ABD$ then

$$\overrightarrow{O_1H_1}=\overrightarrow{O_1A}+\overrightarrow{O_1B}+\overrightarrow{O_1C}\ \text{ and}$$

$$\overrightarrow{O_2H_2}=\overrightarrow{O_2A}+\overrightarrow{O_2B}+\overrightarrow{O_2D}$$

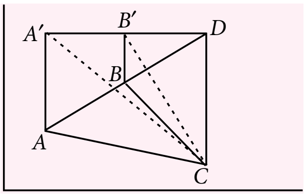
So, $\overrightarrow{O_1H_1}-\overrightarrow{O_2H_2}=2\overrightarrow{O_1O_2}+\overrightarrow{O_1C}-\overrightarrow{O_2D}$

$\Rightarrow \overrightarrow{O_1H_1}-(\overrightarrow{O_2O_1}+\overrightarrow{O_1H_2})=2\overrightarrow{O_1O_2}+\overrightarrow{O_1C}-(\overrightarrow{O_2O_1}+\overrightarrow{O_1D})$

or, $\overrightarrow{H_2H_1}=2\overrightarrow{O_1O_2}+\overrightarrow{DC}$

Hence if $O_1 = O_2$, then $\overrightarrow{H_1H_2}=\overrightarrow{DC}$

**11. (c) :** Notice that $BB'$ is the midline in $\Delta AA'D$, so $AB = BD$. In $\Delta CAD$, the median $CB$ is half of $AD$, *i.e.* $B$ is equidistant from vertices $A$, $C$ and $D$.
Hence $\angle ACD = 90°$




**12. (a) :** Let $A(0, 0, 0)$, $B$ on $x$-axis, $D$ on $y$-axis and $C$ in $xy$-plane and $\angle CBD = \theta$. Let $AB = CD = r$, $BC = AD = 2r$, then

$$\tan\theta = \frac{CD}{BC} = \frac{r}{2r} = \frac{1}{2}$$

Let $N$ be the foot of perpendicular from $C'$

then $\sin\theta = \frac{CN}{BC} \Rightarrow \frac{1}{\sqrt{5}} = \frac{CN}{2r}$

So, $C'N = CN = \frac{2r}{\sqrt{5}}$

Equation of $BD$ is $2x + y = 2r$, $C(r, 2r)$
So, $N$ (foot of $\perp^r$) has co-ordinates

$$\frac{x-r}{2} = \frac{y-2r}{1} = \frac{-(2r+2r-2r)}{5}$$

*i.e.* $x = \frac{r}{5}$, $y = \frac{8r}{5}$

So, $C' \equiv \left(\frac{r}{5}, \frac{8r}{5}, \frac{2r}{\sqrt{5}}\right)$. Hence, $AC' = \frac{\sqrt{85}}{5}r$

**13. (a) :** Notice that $\Delta BDC$ is right angled at $D$ and $\Delta BAC$ is right angled at $A$.
So, if $M$ is the midpoint of $BC$ then

$MA = MB = MC = MD$

**14. (b) :** Notice that the given expression is equivalent to

$(\cos\alpha - \cos\beta\cos\gamma)^2 - \sin^2\beta\sin^2\gamma$ ... (1)

Now, $\alpha, \beta, \gamma \in (0, \pi)$, so, from Euler's inequality of plane angles of a tetrahedral angle, we have

$|\beta - \gamma| < \alpha < (\beta + \gamma)$

*i.e.* $\cos(\beta + \gamma) < \cos\alpha < \cos(\beta - \gamma)$
*i.e.*, $-\sin\beta\sin\gamma < \cos\alpha - \cos\beta\cos\gamma < \sin\beta\sin\gamma$
*i.e.* $(\cos\alpha - \cos\beta\cos\gamma)^2 < \sin^2\beta\sin^2\gamma$
Hence, the relation (1) is less than zero.

**15. (a) :** Let $O$ be the intersection point of the diagonals of the rhombus.

Let $\frac{AM}{AB} = m$, $\frac{BN}{BC} = n$, $\frac{DP}{DC} = p$

Now, $\overrightarrow{OM} = (1-m)\overrightarrow{OA} + m\,\overrightarrow{OB}$

$\overrightarrow{ON} = (1-n)\overrightarrow{OB} + n\,\overrightarrow{OC}$ and

$\overrightarrow{OP} = (1-p)\overrightarrow{OD} + p\,\overrightarrow{OC}$

Let $G$ be the centroid of $\Delta MNP$, then

$$3\,\overrightarrow{OG} = \overrightarrow{OM} + \overrightarrow{ON} + \overrightarrow{OP}$$
$$= (m+n+p-1)\,\overrightarrow{OC} + (m-n+p)\,\overrightarrow{OB}$$

[As $\overrightarrow{OA} = -\overrightarrow{OC}$, $\overrightarrow{OD} = -\overrightarrow{OB}$]

Now, $G$ lies on $AC$ if $\overrightarrow{OC}$ and $\overrightarrow{OG}$ are collinear
*i.e.* $m - n + p = 0$ *i.e.* $m + p = n$
or $AM + DP = BN$

**16. (a) :** Let $\overrightarrow{AB} = x$, $\overrightarrow{AC} = y$ and $\overrightarrow{AD} = z$

then $\overrightarrow{MN} = \frac{y+z-x}{2}$

and since $\overrightarrow{MN}\cdot\overrightarrow{AB} = 0$, we have $xz + xy - x^2 = 0$

and $\overrightarrow{MN}\cdot\overrightarrow{CD} = 0$, we have $y^2 - z^2 - xy + xz = 0$

Adding, we have $y^2 = (x - z)^2$ *i.e.* $AC = BD$

**17. (c) :** Area of $\Delta ABC = a^2\cdot\sqrt{3}$
and area of $\Delta ABC = \cos\alpha \times (\Delta MNC)$
where $\alpha = 30°$ is the angle between planes $ABC$ and $MNC$

So, area of $\Delta MNC = \frac{\sqrt{3}a^2}{\cos 30°} = 2a^2$.

**18. (a) :** Let $O(0, 0, 0)$ and edges be along co-ordinate axes. Further let, $A(a, -b, -c)$, $B(a, b, c)$, $C(-a, b, -c)$
So, $\cos\alpha + \cos\beta + \cos\theta$

$$= \frac{(a^2-b^2-c^2)+(-a^2+b^2-c^2)+(-a^2-b^2+c^2)}{a^2+b^2+c^2}$$

$= -1.$

**19. (b) :** The projection will be an ellipse with major axis $= 2a$
Let minor axis $= b$ then area projection formula gives

$\pi ab = \cos\theta \times \pi a^2$ *i.e.*, $b = a\cos\theta$

So, eccentricity, $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1-\cos^2\theta} = \sin\theta$

◇◇

## MPP-6 CLASS XII ANSWER KEY

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **1.** | (b) | **2.** | (d) | **3.** | (c) | **4.** | (c) | **5.** | (c) |
| **6.** | (c) | **7.** | (a,b) | **8.** | (b,c,d) | **9.** | (b,c) | **10.** | (b,d) |
| **11.** | (a) | **12.** | (a,d) | **13.** | (a,d) | **14.** | (a) | **15.** | (a) |
| **16.** | (b) | **17.** | (3) | **18.** | (2) | **19.** | (4) | **20.** | (1) |

# MPP-6 MONTHLY Practice Problems


This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

## Definite Integration & Application of Integrals

**Total Marks : 80** **Time Taken : 60 Min.**

### Only One Option Correct Type

**1.** $\displaystyle\int_0^{\pi/2} \frac{1+2\cos x}{(2+\cos x)^2}dx =$

(a) 2 (b) $\frac{1}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{1}{4}$

**2.** Let $f(x)+f\left(\frac{1}{x}\right)=F(x)$, where $f(x)=\displaystyle\int_1^x \frac{\ln t}{1+t}dt$. Then $F(e)$ =

(a) 0 (b) 1 (c) 2 (d) $\frac{1}{2}$

**3.** The area (in sq. units) bounded by the curves $y=x^2$ and $y=\dfrac{2}{1+x^2}$ is

(a) $\pi$ (b) $\pi-\frac{1}{3}$ (c) $\pi-\frac{2}{3}$ (d) $\pi+\frac{2}{3}$

**4.** If $f(x)=\displaystyle\int_0^x \sin^4 t\,dt$, then $f(x+\pi)$ is equal to

(a) $f(\pi)$ (b) $f(x)$
(c) $f(x)+f(\pi)$ (d) $f(x)\cdot f(\pi)$

**5.** Area bounded by $|x-1|\le 2$ and $x^2-y^2=1$, (in sq. units) is

(a) $6\sqrt{2}+\frac{1}{2}\log|3+2\sqrt{2}|$ (b) $6\sqrt{2}+\frac{1}{2}\log|3-2\sqrt{2}|$
(c) $6\sqrt{2}-\log|3+2\sqrt{2}|$ (d) None of these

**6.** $\displaystyle\int_{-3\pi/2}^{-\pi/2} [(x+\pi)^3+\cos^2(x+3\pi)]dx$ is equal to

(a) $\frac{\pi^4}{32}$ (b) $\frac{\pi^4}{32}+\frac{\pi}{2}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{2}-1$

### One or More Than One Option(s) Correct Type

**7.** If $f(x)=A\cdot 2^x+B$, where $f'(1)=2$ and $\displaystyle\int_0^3 f(x)dx=7$, then

(a) $A=\dfrac{1}{\log 2}$

(b) $B=\dfrac{7}{3(\log 2)^2}[(\log 2)^2-1]$

(c) $A=\dfrac{7}{3(\log 2)^2}[(\log 2)^2-1]$

(d) $B=\dfrac{1}{\log 2}$

**8.** Area of the region bounded by the curve $y=e^x$ and lines $x=0$ and $y=e$ (in sq. units) is

(a) $e-1$ (b) $\displaystyle\int_1^e \ln(e+1-y)\,dy$

(c) $e-\displaystyle\int_0^1 e^x\,dx$ (d) $\displaystyle\int_1^e \ln y\,dy$

**9.** Area bounded by the curve $y=\ln x$, $y=0$ and $x=3$ is

(a) $(\ln 9-2)$ sq. units (b) $(\ln 27-2)$ sq. units

(c) $\ln\left(\frac{27}{e^2}\right)$ sq. units (d) (greater than 3) sq. units

**10.** For which of the following values of $m$, is the area of the region bounded by the curve $y=x-x^2$ and the line $y=mx$ equals 9/2 sq. units?

(a) –4 (b) –2 (c) 2 (d) 4

**11.** The value(s) of $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$ is (are)

(a) $\frac{22}{7}-\pi$ (b) $\frac{2}{105}$ (c) 0 (d) $\frac{71}{15}-\frac{3\pi}{2}$

**12.** Let $f(x)=\int_0^{\sin^2 x}\sin^{-1}(\sqrt{t})dt+\int_0^{\cos^2 x}\cos^{-1}(\sqrt{t})\,dt$, then

(a) $f(x)$ is a constant function

(b) $f\left(\frac{\pi}{4}\right)=0$ (c) $f\left(\frac{\pi}{3}\right)=\pi$

(d) $f\left(\frac{\pi}{4}\right)=\frac{\pi}{4}$

**13.** Area bounded by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ equals

(a) $6\pi$ sq. units

(b) $3\pi$ sq. units

(c) $12\pi$ sq.units

(d) area bounded by the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$

## Comprehension Type

Suppose we define the definite integral using the following formula $\int_a^b f(x)dx=\frac{b-a}{2}(f(a)+f(b))$, for more accurate result for $c\in(a, b)$,

$F(c)=\frac{c-a}{2}(f(a)+f(c))+\frac{b-c}{2}(f(b)+f(c))$

When

$c=\frac{a+b}{2}, \int_a^b f(x)dx=\frac{b-a}{4}(f(a)+f(b)+2f(c)).$

**14.** $\int_0^{\pi/2}\sin x\,dx$ is equal to

(a) $\frac{\pi}{8}\left(1+\sqrt{2}\right)$ (b) $\frac{\pi}{4}\left(1+\sqrt{2}\right)$

(c) $\frac{\pi}{8\sqrt{2}}$ (d) $\frac{\pi}{4\sqrt{2}}$

**15.** If $f(x)$ is a polynomial and if

$$\lim_{t\to a}\frac{\int_a^t f(x)dx-\frac{(t-a)}{2}(f(t)+f(a))}{(t-a)^3}=0$$

for all $a$, then the degree of $f(x)$ can atmost be

(a) 1 (b) 2 (c) 3 (d) 4

## Matrix Match Type

**16.** Match the following:

| | Column I | Column II | |
|---|---|---|---|
| P. | $\int_0^\infty e^{-4x}\sin 5x dx=$ | 1. | 3 |
| Q. | $\int_2^8\frac{[x^2]dx}{[x^2-20x+100]+[x^2]}=$ (where $[\cdot]$ is greater integer function) | 2. | $\frac{5}{41}$ |
| R. | $\int_0^{3\pi/2}\lvert\sin x\rvert dx$, (where $n\in N$)= | 3. | 120 |
| S. | $\int_0^\infty x^5e^{-x}dx=$ | 4. | 60 |

| | P | Q | R | S |
|---|---|---|---|---|
| (a) | 2 | 1 | 4 | 3 |
| (b) | 2 | 1 | 1 | 3 |
| (c) | 2 | 4 | 1 | 3 |
| (d) | 1 | 2 | 4 | 3 |

## Integer Answer Type

**17.** Let $n\in N$, $n\le 5$.

If $I_n=\int_0^1 e^x(x-1)^n dx=16-6e$, then $n=$

**18.** The value of $\int_0^\infty\left[\frac{2}{e^x}\right]dx$ is $\log_e a$ (where $[\cdot]$ is greatest integer function), then $a$ is equal to

**19.** The area bounded by the curves $x=y^2$ and $x=3-2y^2$ (in sq. units) is

**20.** If $a$ is a positive integer, then number of values of $a$ satisfying

$$\int_0^{\pi/2}\left\{a^2(\sin 3x+2\cos x)+\frac{a}{3}\sin x-2\cos x\right\}dx\le\frac{2a^2}{3}$$

are

*Keys are published in this issue. Search now!* ☺

**SELF CHECK**

No. of questions attempted ......

No. of questions correct ......

Marks scored in percentage ......

**Check your score! If your score is**

| | | |
|---|---|---|
| > 90% | EXCELLENT WORK ! | You are well prepared to take the challenge of final exam. |
| 90-75% | GOOD WORK ! | You can score good in the final exam. |
| 74-60% | SATISFACTORY ! | You need to score more next time. |
| < 60% | NOT SATISFACTORY! | Revise thoroughly and strengthen your concepts. |

# MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

## PROBLEM Set 168

### JEE MAIN

**1.** The difference of the roots of the equation

$(x^2 - 10x - 29)^{-1} + (x^2 - 10x - 45)^{-1} = 2(x^2 - 10x - 69)^{-1}$ is

(a) 3 (b) –3 (c) 13 (d) 16

**2.** Ten boys and two girls are to be seated in a row such that there are atleast 3 boys between the girls. The number of ways this can be done is $\lambda \cdot 12!$, where $\lambda =$

(a) 2/3 (b) 4/11 (c) 5/11 (d) 6/11

**3.** In a triangle $ABC$ if $a, b, c$ are in A.P. and $C - A = 120°$, then $\frac{s}{r} =$

(a) $\sqrt{15}$ (b) $2\sqrt{15}$ (c) $3\sqrt{15}$ (d) $4\sqrt{15}$

**4.** If the lines $ax - 2y + 2 = 0$ and $2x - y - 3 = 0$ meet the coordinate axes at concyclic points, then the length of the segment of the line $y = x$ made by the circle through the four points is

(a) $\frac{11}{\sqrt{2}}$ (b) $\frac{11}{2}$ (c) $\frac{11}{2\sqrt{2}}$ (d) $\frac{11}{4}$

**5.** If $P(x) = x^6 - x^5 - x^3 - x^2 - x$ and $\alpha, \beta, \gamma, \delta$ are the roots the equation $Q(x) = x^4 - x^3 - x^2 - 1 = 0$, then $P(\alpha) + P(\beta) + P(\gamma) + P(\delta) =$

(a) 6 (b) 8 (c) 10 (d) 12

### JEE ADVANCED

**6.** The vertices of a triangle $ABC$ are $A(1, 0)$ while $B$ and $C$ lie on the parabola $y = 2x - x^2$.

If $AB = AB = AC = \frac{\sqrt{7}}{3}$, then its area (in sq. units)is

(a) $\frac{1}{3\sqrt{3}}$ (b) $\frac{1}{3}\sqrt{\frac{2}{3}}$ (c) $\frac{2}{3\sqrt{3}}$ (d) $\frac{2}{3}$

### COMPREHENSION

Let $f(x)$ be a differentiable function such that $f(1) = 0$ and $2\int_0^x f(t)dt - \int_0^1 f(t)dt = 2f(x) + 3x + a$, where $a$ is a constant. Then,

**7.** $a =$

(a) $\frac{3}{2e} - 3$ (b) $\frac{3}{2e} - \frac{3}{2}$

(c) $\frac{3}{2e} + 3$ (d) $\frac{3}{2e} + \frac{3}{2}$

**8.** $f'(1) =$

(a) – 3 (b) $-\frac{3}{2}$ (c) – 2 (d) – 1

### INTEGER MATCH

**9.** Let $|\vec{a}| = 1, |\vec{b}| = 4$ and $\vec{a} \times \vec{r} + \vec{b} = \vec{r}$. If projection of $\vec{r}$ along $\vec{a}$ is 2, then the projection of $\vec{r}$ along $\vec{b}$ is $\frac{m}{n}$ in the reduced form, where $m - n$ is

### MATRIX MATCH

**10.** Match the following:

| | Column I | Column II | |
|---|---|---|---|
| P. | If the fourth term in the expansion of $\left(\frac{x}{a} + \frac{1}{x}\right)^n$ is $\frac{5}{2}$, then $a$ is divisible by | 1. | 3 |
| Q. | The coefficient of $x^{13}$ in $(1 - x)^5 \ (1 + x + x^2 + x^3)^4$ is divisible by | 2. | 16 |
| R. | $\sum_{k=0}^{4} \binom{4}{k}(k-2)^2 =$ | 3. | 13 |
| S. | $\sum_{p=1}^{4} \sum_{r=p}^{4} \binom{4}{r}\binom{r}{p}$ is divisible by | 4. | 2 |

| | P | Q | R | S |
|---|---|---|---|---|
| (a) | 1 | 2 | 3 | 4 |
| (b) | 4 | 4 | 2 | 3 |
| (c) | 3 | 1 | 2 | 4 |
| (d) | 1 | 3 | 2 | 4 |

See Solution Set of Maths Musing 167 on page no. 85

# WB JEE MOCK TEST PAPER

**Series-5**

**The entire syllabus of Mathematics of WB-JEE is being divided in to eight units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issues. The syllabus for module break-up is given bellow.**

| Unit No. | Topic | Syllabus In Details |
|---|---|---|
| UNIT NO. 5 | Differential Calculus | Functions, Limits, Continuity & Differentiability. |
| | Statistics & Probability | Measures of Dispersion: Calculation of mean, median, mode of grouped and ungrouped data, calculation of standard deviation, variance and mean deviation for grouped and ungrouped data.<br>Probability: Probability of an event, addition and multiplication theorems of probability. |
| | Co-ordinate Geometry-3D | Coordinate axes and coordinate planes in three dimensions. Coordinate of a point. Distance between two points and section formula. |

**Time : 1 hr 15 min.** **Full marks : 50**

## CATEGORY-I

**For each correct answer one mark will be awarded, whereas, for each wrong answer, 25% of total marks (1/4) will be deducted. If candidates mark more than one answer, negative marking will be done.**

**1.** If $f(x)=\lim\limits_{t\to\infty}\dfrac{(1+\sin\pi x)^t-1}{(1+\sin\pi x)^t+1}$, then range of $f(x)$ is

(a) $\{-1, 1\}$ (b) $\{0, 1\}$
(c) $[-1, 1]$ (d) $\{-1, 0, 1\}$

**2.** The graph of $y = f(x)$ is symmetrical about the line $x = 1$, then

(a) $f(-x) = f(x)$
(b) $f(1 + x) = f(1 - x)$
(c) $f(x + 1) = f(x - 1)$
(d) none of these

**3.** If $f(x) = 2x^n + a$, if $f(2) = 26$ and $f(4) = 138$, then $f(3) =$

(a) 56 (b) 112 (c) 82 (d) 64

**4.** The domain of $f(x)=\sqrt{\log_3\cos(\sin x)}$ is

(a) $x = n\pi$ (b) $x=\dfrac{n\pi}{2}$
(c) $x = \phi$ (d) none of these

**5.** A polynomial function $f(x)$ satisfies the condition $f(x)\cdot f\left(\dfrac{1}{x}\right)=f(x)+f\left(\dfrac{1}{x}\right)$. If $f(10) = 1001$, $f(20) =$

(a) 8001 (b) 8008
(c) 8002 (d) none of these

**6.** Let $f(x) = \log_e x + \log_x e$, then the domain of the function $\dfrac{1}{\sqrt{|f(x)|-f(x)}}$ is

(a) $(0, 1)$ (b) $(1, \infty)$
(c) $(1, e)$ (d) $(-\infty, \infty)$

**7.** If $\alpha$, $\beta$ are the roots of $x^2 - ax + b = 0$, then $\lim\limits_{x\to\alpha}\dfrac{e^{x^2-ax+b}-1}{x-\alpha}=$

(a) $\beta - \alpha$ (b) $\alpha - \beta$
(c) 1 (d) $2\alpha - a$

**8.** $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cot x-\cos x}{(\pi-2x)^3}=$

(a) 1/16 (b) 1 (c) 1/8 (d) $\infty$

**9.** $\lim\limits_{x\to 3}\dfrac{1-\cos(x^2-x-6)}{(x-3)^2}=$

By : Sankar Ghosh, S.G.M.C, Kolkata, Ph: 09831244397.

(a) $-\frac{1}{2}$ (b) $\frac{25}{2}$ (c) $\frac{1}{2}$ (d) $-\frac{25}{2}$

**10.** If $f(x)=\begin{cases} -1 & \text{for} \quad x<-1 \\ x^3 & \text{for} \quad -1\le x\le 1 \\ 1-x & \text{for} \quad 1<x<2 \\ 3-x^2 & \text{for} \quad x\ge 2 \end{cases}$

Then $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$ equal to

(a) 1, –1 (b) –1, –1
(c) 0, 1 (d) 0, –1

**11.** $\lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x}=e^2$, then

(a) $a = 1, b = 0$ (b) $a = 2, b = 1$
(c) $a = 1, b \in R$ (d) $a = 1, b = 1$

**12.** If $f: R \to R$ be a differentiable function, such that $f(x + 2y) = f(x) + f(2y) + 4xy \ \forall\ x, y \in R$, then

(a) $f'(1) = f'(0) + 1$
(b) $f'(1) = f'(0) - 1$
(c) $f'(0) = f'(1) + 2$
(d) $f'(0) = f'(1) - 2$

**13.** If $f(x)=\begin{cases} \dfrac{\sin\{\cos x\}}{x-\frac{\pi}{2}}, & x\ne\frac{\pi}{2} \\ 1, & x=\frac{\pi}{2} \end{cases}$

where $\{\cdot\}$ represents the fractional part, then

(a) $f(x)$ is continuous at $x = \pi/2$
(b) $\lim_{x\to\pi/2} f(x)$ exists, but $f$ is not continuous at $x = \pi/2$
(c) $\lim_{x\to\pi/2} f(x)$ does not exist
(d) $\lim_{x\to\pi/2} f(x)=-1$

**14.** If $f(x)=\begin{cases} \dfrac{x-[x]}{x-3}, & x<4 \\ b, & x=4 \\ \dfrac{a|x^2-5x+6|}{(x-3)}, & x>4 \end{cases}$,

is continuous at $x = 4$, then

(a) $a = 0, b = 1$
(b) $a = 1/2, b = 1$
(c) $a = -1/2, b = 1$
(d) $f(x)$ is continuous everywhere for any real $a$ and $b$

**15.** If the function $f(x)=\begin{cases} (1+|\sin x|)^{\frac{a}{|\sin x|}}, & -\frac{\pi}{6}<x<0 \\ b, & x=0 \\ e^{\frac{\tan 2x}{\tan 3x}}, & 0<x<\frac{\pi}{6} \end{cases}$

is continuous at $x = 0$, then

(a) $a=\log_e b, a=\frac{2}{3}$ (b) $b=\log_e a, a=\frac{2}{3}$
(c) $a = \log_e b, b = 2$ (d) none of these

**16.** For a differentiable function $f$,

$$\lim_{h\to 0}\frac{\{f(x+h)\}^2-\{f(x)\}^2}{2h}=$$

(a) $\{f'(x)\}^2$ (b) $\frac{1}{2}\{f'(x)\}^2$
(c) $\frac{1}{2}[\{f'(x)\}^2-\{f(x)\}^2]$
(d) $f(x)\,f'(x)$

**17.** A boy goes to school from his home at a speed of $x$ km/h and comes back at a speed of $y$ km/h; then his average speed is

(a) A.M. of $x$ and $y$ (b) H.M. of $x$ and $y$
(c) G.M. of $x$ and $y$ (d) none of these

**18.** The A.M. of 7, $x - 3$, 10, $x + 3$, and $x - 5$ is 15. The median of the numbers is

(a) 21 (b) 16
(c) 17 (d) none of these

**19.** If the mode and A.M. are ₹ 12.30 and ₹ 18.48 respectively, then median of the distribution (in ₹) is

(a) 16.42 (b) 12.30
(c) 15.39 (d) none of these

**20.** The standard deviation of 50 values of a variable $x$ is 15; if each value of the variable is divided by (–3); then the standard deviation of the new set of 50 values of $x$ will be

(a) 15 (b) –5 (c) 5 (d) –15

**21.** If the sum of the squares of the deviations of 25 observations taken from the mean 40 is 900, then the coefficient of variation is

(a) 20% (b) 12.5%
(c) 15% (d) 18%

**22.** A coin and a six faced die both unbiased are thrown simultaneously. The probability of getting a head on the coin and an odd number on the die is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

**23.** A sample of 4 items is drawn at random from a lot of 10 items, containing 3 defectives. If $x$ denotes the number of defective items in the sample, then $P(0 < x < 3)$ is equal to

(a) $\frac{3}{10}$ (b) $\frac{1}{2}$ (c) $\frac{4}{5}$ (d) $\frac{1}{6}$

**24.** The probability that in a family of 5 members, exactly 2 members have birthday on Sunday is

(a) $\frac{12\times 5^3}{7^5}$ (b) $\frac{10\times 6^2}{7^5}$

(c) $\frac{2}{3}$ (d) $\frac{10\times 6^3}{7^5}$

**25.** A five digit number is formed by writing the digits 1, 2, 3, 4, 5 in a random order without repetition. Then the probability that the number is divisible by 4 is

(a) $\frac{3}{5}$ (b) $\frac{1}{5}$ (c) $\frac{5}{6}$ (d) $\frac{5}{18}$

**26.** The co-ordinates of the foot of the perpendicular drawn from the point $P(x, y, z)$ upon the $zx$-plane are

(a) $(x, 0, 0)$ (b) $(0, 0, z)$

(c) $(x, y, 0)$ (d) $(x, 0, z)$

**27.** The co-ordinates of the point of trisection of the line-segment joining the points (2, 1, –3) and (5, –8, 3) that is nearer to (5, –8, 3) are

(a) (4, 5, –1) (b) (3, –2, –1)

(c) (2, –1, 3) (d) (4, –5, 1)

**28.** $C$ is a point on the line segment joining the points $A(2, -3, 4)$ and $B(8, 0, 10)$; if the $y$-co-ordinate of $C$ is (–2), then its $z$-co-ordinate will be

(a) 4 (b) 6 (c) –4 (d) 5

**29.** The ratio in which the straight line joining the points $A(3, 5, -7)$ and $B(-2, 1, 8)$ is divided by $yz$-plane is

(a) 3 : 2 (b) 2 : 3 (c) –3 : 4 (d) 1 : 2

**30.** The co-ordinate of the vertices $B$ and $C$ of the triangle $ABC$ are (5, 2, 8) and (2, –3, 4) respectively; if the co-ordinates of the centroid of the triangle are (3, –1, 3), then the co-ordinates of the vertex $A$ are

(a) (2, –2, 2) (b) (2, –2, –3)

(c) (2, 2, –3) (d) (–2, –2, –3)

## CATEGORY-II

**Every correct answer will yield 2 marks. For incorrect response, 25% of full mark (1/2) would be deducted. If candidates mark more than one answer, negative marking will be done.**

**31.** The domain of definition of the function $f(x)=\sqrt{\log_e(x^2-6x+6)}$ is

(a) $(-\infty, 3-\sqrt{3}] \cup [3+\sqrt{3}, \infty)$

(b) $(-\infty, 1] \cup [5, \infty)$

(c) $(-\infty, 3-\sqrt{3}) \cup (3+\sqrt{3}, \infty)$

(d) $(-\infty, 1) \cup (5, \infty)$

**32.** In a group of 20 males and 5 females, 10 males and 3 females are service holders. The probability that a person selected at random from the group, is a service holder, given that the selected person is a male is

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{2}{5}$ (d) $\frac{3}{5}$

**33.** Two candidates $A$ and $B$ appeared for an interview for two vacancies. The probabilities of their selection are $\frac{1}{4}$ and $\frac{1}{3}$ respectively. The probability that one of them will be selected is

(a) $\frac{3}{4}$ (b) $\frac{2}{3}$ (c) $\frac{7}{12}$ (d) $\frac{5}{12}$

**34.** The value of $\lim\limits_{x\to 4}\dfrac{x^{\frac{7}{2}}-4^{\frac{7}{2}}}{\log_e(x-3)}$ is

(a) 112 (b) 80 (c) 96 (d) 56

**35.** The function $f(x) = p[x + 1] + q[x - 1]$ where $[\cdot]$ is the greatest integer function and $\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^-} f(x)=f(1)$ when

(a) $p = 0$ (b) $q = 0$

(c) $p + q = 0$ (d) $p - q = 0$

## CATEGORY-III

**In this section more than 1 answer can be correct. Candidates will have to mark all the correct answers, for which 2 marks will be awarded. If, candidates marks one correct and one incorrect answer then no marks will be awarded. But if, candidate makes only correct, without making any incorrect, formula below will be used to allot marks.**

**2 × (no. of correct response/total no. of correct options)**

**36.** Let $f(x) = [x]$, the greatest integer function less than or equal to $x$ and $g(x) = x - [x]$. Then for two real numbers $x$ and $y$

(a) $f(x + y) = f(x) + f(y)$

(b) $g(x + y) = g(x) + g(y)$

(c) $f(x + y) = f(x) + f(y + g(x))$

(d) none of these

**37.** If $\bar{E}$ and $\bar{F}$ are the complimentary events of $E$ and $F$ respectively, then

(a) $P(E/F)+P(\bar{E}/F)=1$

(b) $P(E/F)+P(E/\bar{F})=1$

(c) $P(\bar{E}/F)+P(E/\bar{F})=1$

(d) $P(E/\bar{F})+P(\bar{E}/\bar{F})=1$

**38.** Let $A$ and $B$ be two events such that $P(A\cap B)=\frac{1}{3}$, $P(A\cup B)=\frac{5}{6}$ and $P(\bar{A})=\frac{1}{2}$. Then

(a) $A$, $B$ are independent

(b) $A$, $B$ are mutually exclusive

(c) $P(A) = P(B)$ (d) $P(B) \le P(A)$

**39.** If $\alpha$ is a repeated root of $ax^2 + bx + c = 0$ then $\lim\limits_{x\to\alpha}\frac{\sin(ax^2+bx+c)}{(x-\alpha)^2}$ is

(a) 0 (b) $a$ (c) $b$ (d) $c$

**40.** If $f(|x - 1| - [x])$, where $[x]$ is the greatest integer function less than or equal to $x$, then

(a) $f(1^+) = -1, f(1^-) = 0$

(b) $f(1^+) = 0 = f(1^-)$

(c) $\lim\limits_{x\to1} f(x)$ exists (d) $\lim\limits_{x\to1} f(x)$ does not exists

## SOLUTIONS

**1. (d) :** $f(x)=\lim\limits_{t\to\infty}\frac{(1+\sin\pi x)^t-1}{(1+\sin\pi x)^t+1}$

$$=\lim_{t\to\infty}\frac{1-\frac{1}{(1+\sin\pi x)^t}}{1+\frac{1}{(1+\sin\pi x)^t}}=\begin{cases}1 \text{ when } \sin\pi x>0\\ -1 \text{ when } \sin\pi x<0\\ 0 \text{ when } \sin\pi x=0\end{cases}$$

$\therefore$ Range of $f(x) = \{-1, 0, 1\}$

**2. (b) :** Graph of $y = f(x)$ is symmetrical about the line $x = 0$, if $f(-x) = f(x)$ *i.e.* if $f(0 - x) = f(0 + x)$.

$\therefore$ Graph of $y = f(x)$ is symmetrical about $x = 1$, if $f(1 - x) = f(1 + x)$.

**3. (d) :** Given that

$f(x) = 2x^n + a$. And $f(2) = 26, f(4) = 138$

$\therefore$ $f(2) = 2\cdot 2^n + a = 26$ ... (1)

And, $f(4) = 2\cdot 4^n + a = 138$ ... (2)

From (1) and (2) we get, $2\cdot 4^n - 2\cdot 2^n = 112$

$\Rightarrow (2^n)^2 - 2^n - 56 = 0$

$\Rightarrow (2^n - 8)(2^n + 7) = 0 \Rightarrow n = 3$

$\therefore$ $f(x) = 2x^3 + a$

Now from (1) we get, $2\cdot 2^3 + a = 26 \Rightarrow a = 10$

$\therefore$ $f(x) = 2x^3 + 10$. Thus, $f(3) = 2\cdot 3^3 + 10 = 64$

**4. (a) :** In order that $f(x)$ be defined $\log_3\cos(\sin x) \ge 0$

$\Rightarrow \cos(\sin x) \ge 1 \Rightarrow \cos(\sin x) = 1$ $(\because -1 \le \cos\theta \le 1)$

$\Rightarrow \sin x = 0 \Rightarrow x = n\pi, n \in I$

**5. (a) :** Let $f(x)=a_0x^n+a_1x^{n-1}+...+a_n$.

Since, $f(x)\cdot f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)$ we get,

$$(a_0x^n+a_1x^{n-1}+...+a_n)\cdot\left(\frac{a_0}{x^n}+\frac{a_1}{x^{n-1}}+...+a_n\right)$$

$$=(a_0x^n+a_1x^{n-1}+...+a_n)+\left(\frac{a_0}{x^n}+\frac{a_1}{x^{n-1}}+...+a_n\right)$$

On comparison of co-efficient of like powers of $x$ we get $a_0 = \pm1$ and $a_n = 1$ and $a_1 = a_2 = ... = a_{n-1} = 0$

$\Rightarrow f(x) = x^n + 1, f(10) = 1001 = 10^n + 1 \Rightarrow n = 3$

$\Rightarrow f(x) = x^n + 1, f(20) = 20^3 + 1 = 8001$

**6. (a) :** The function $\frac{1}{\sqrt{|f(x)|-f(x)}}$ will be defined when $|f(x)| - f(x) > 0$

$\Rightarrow |f(x)| > f(x) \geq f(x) < 0$

$\Rightarrow \log_e x + \log_x e < 0$ $(\because f(x) = \log_e x + \log_x e)$

$\Rightarrow \log_e x+\frac{1}{\log_e x}<0 \Rightarrow \frac{(\log_e x)^2+1}{\log_e x}<0$

$\Rightarrow \log_e x < 0 \Rightarrow 0 < x < 1$

**7. (b) :** $\lim\limits_{x\to\alpha}\frac{e^{x^2-ax+b}-1}{x-\alpha}$

$$=\lim_{x\to\alpha}\frac{e^{(x-\alpha)(x-\beta)}-1}{(x-\alpha)(x-\beta)}\times(x-\beta)\qquad\left[\because \lim_{y\to0}\frac{e^y-1}{y}=1\right]$$

$= \alpha - \beta$

**8. (a) :** $\lim\limits_{x\to\frac{\pi}{2}}\frac{\cot x-\cos x}{(\pi-2x)^3}=\lim\limits_{x\to\frac{\pi}{2}}\frac{\cot x(1-\sin x)}{(\pi-2x)^3}$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\tan\left(\frac{\pi}{2}-x\right)2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{(\pi-2x)^3}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\tan\left(\frac{\pi}{2}-x\right)}{2\left(\frac{\pi}{2}-x\right)}\cdot2\frac{\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{16\left(\frac{\pi}{4}-\frac{x}{2}\right)^2}=\frac{1}{2}\cdot\frac{2}{16}=\frac{1}{16}$$

**9. (b) :** $\lim_{x\to 3}\frac{1-\cos(x^2-x-6)}{(x-3)^2}=\lim_{x\to 3}\frac{1-\cos\{(x-3)(x+2)\}}{(x-3)^2}$

$=\lim_{h\to 0}\frac{2\sin^2\frac{h(h+5)}{2}}{h^2}$ (let $x-3=h$)

$=\lim_{h\to 0}\frac{2\sin^2\frac{h(h+5)}{2}}{4\left\{\frac{h(h+5)}{2}\right\}^2}\times(h+5)^2=\frac{1}{2}\cdot 5^2=\frac{25}{2}$

**10. (b) :** $\lim_{x\to 2^+}f(x)=\lim_{h\to 0}3-(2+h)^2=3-4=-1$

$\lim_{x\to 2^-}f(x)=\lim_{h\to 0}1-(2-h)=1-2=-1$

**11. (c) :** $\lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x}=\lim_{x\to\infty}\left(1+\frac{ax+b}{x^2}\right)^{2x}$

$=\lim_{x\to\infty}\left(1+\frac{1}{\frac{x^2}{ax+b}}\right)^{\frac{x^2}{ax+b}\times\frac{2(ax+b)}{x}}$

$=e^{\lim_{x\to\infty}\frac{2(ax+b)}{x}}$ $\left(\because \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e\right)$ $=e^{2a}$

By the problem, $e^{2a}=e^2 \Rightarrow a=1, b\in R$

**12. (d) :** We have,

$f(x+2y)=f(x)+f(2y)+4xy, \forall\ x, y\in R$

$\Rightarrow \frac{f(x+2y)-f(x)}{2y}=2x+\frac{f(2y)}{2y}$

$\Rightarrow \lim_{2y\to 0}\frac{f(x+2y)-f(x)}{2y}=\lim_{2y\to 0}\left(2x+\frac{f(2y)}{2y}\right)$

$\Rightarrow f'(x)=2x+f'(0)$ [as $f(0)=0$]

$\Rightarrow f'(0)=f'(1)-2$

**13. (c) :** We have, $f(x)=\begin{cases}\frac{\sin\{\cos x\}}{x-\frac{\pi}{2}}, & x\ne\frac{\pi}{2}\\ 1, & x=\frac{\pi}{2}\end{cases}$,

where $\{\cdot\}$ represents the fractional part.

Now, $\lim_{x\to\frac{\pi}{2}^-}f(x)=\lim_{h\to 0}\left(\frac{\sin\{\sin h\}}{-h}\right)=-1$

Again $\lim_{x\to\frac{\pi}{2}^+}f(x)=\lim_{h\to 0}\left(\frac{\sin\{-\sin h\}}{h}\right)\ne -1$

$\therefore \lim_{x\to\frac{\pi}{2}^-}f(x)\ne\lim_{x\to\frac{\pi}{2}^+}f(x)$

So $\lim_{x\to\pi/2}f(x)$ does not exist.

**14. (b) :** The given function is

$f(x)=\begin{cases}\frac{x-[x]}{x-3}, & x<4\\ b, & x=4\\ \frac{a|x^2-5x+6|}{(x-3)}, & x>4\end{cases}$

Now $\lim_{x\to 4^+}f(x)=\lim_{x\to 4^+}\frac{a|x^2-5x+6|}{(x-3)}$

$=\lim_{x\to 4^+}\frac{a(x-2)(x-3)}{(x-3)}=2a$

Again $\lim_{x\to 4^-}\frac{x-[x]}{x-3}=\frac{4-3}{4-3}=1,$

(when $3\le x<4$, then $[x]=3$)

Since $f(x)$ is continuous at $x=4$

$\therefore\ 2a=b=1 \Rightarrow a=\frac{1}{2}, b=1.$

**15. (a) :** Here the given function is

$f(x)=\begin{cases}(1+|\sin x|)^{\frac{a}{|\sin x|}}, & -\frac{\pi}{6}<x<0\\ b, & x=0\\ e^{\frac{\tan 2x}{\tan 3x}}, & 0<x<\frac{\pi}{6}\end{cases}$

$\lim_{x\to 0^-}f(x)=\lim_{x\to 0^+}f(x)=f(0)$

$\Rightarrow \lim_{x\to 0^-}(1+|\sin x|)^{\frac{a}{|\sin x|}}=\lim_{x\to 0^+}e^{\frac{\tan 2x}{\tan 3x}}=b$

$\Rightarrow e^a=e^{\frac{2}{3}}=b \quad \therefore a=\frac{2}{3}, a=\log_e b$

**16. (d) :** $\lim_{h\to 0}\frac{\{f(x+h)\}^2-\{f(x)\}^2}{2h}$

$=\frac{1}{2}\lim_{h\to 0}\{f(x+h)+f(x)\}\times\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=\frac{1}{2}\cdot 2f(x)f'(x)=f(x)f'(x)$

**17. (b) :** Let the distance between the boy's home and school be $d$ km.

$\therefore$ Time required to reach school $= \frac{d}{x}$ hours

and time required to return to home $= \frac{d}{y}$ hours

Average speed $= \frac{2d}{\frac{d}{x}+\frac{d}{y}}$ km/hours

$= \frac{2xy}{x+y}$ km/hours = H.M. of $x$ and $y$

**18. (b) :** By the problem we have

$$\frac{7+x-3+10+x+x+3+x-5}{5}=15$$

$\Rightarrow \frac{3x+12}{5}=15 \Rightarrow x=21$

$\therefore$ The numbers are 7, 21 – 3, 10, 21 + 3 and 21 – 5 *i.e.* 7, 18, 10, 24 and 16

Now arranging the numbers in ascending order, we get

7, 10, 16, 18, 24.

$\therefore$ The required median is 16

**19. (a) :** The empirical relation between mean, median and mode is

3 Median = Mode + 2Mean

$\Rightarrow$ Median $= \frac{12.30+2\times 18.48}{3} = 16.42$

**20. (c) :** Let $y_i = -\frac{1}{3}x_i$ where $i = 1, 2, ..., 50$ and $\sigma_x = 15$

$\therefore \bar{y} = -\frac{1}{3}\bar{x}$

Now, $y_i - \bar{y} = -\frac{1}{3}(x_i - \bar{x}) \Rightarrow (y_i-\bar{y})^2 = \frac{1}{9}(x_i-\bar{x})^2$

$\Rightarrow \frac{1}{50}\sum_{i=1}^{50}(y_i-\bar{y})^2 = \frac{1}{9}\cdot\frac{1}{50}\sum_{i=1}^{50}(x_i-\bar{x})^2$

$\Rightarrow \sigma_y^2 = \frac{1}{9}\sigma_x^2 \Rightarrow \sigma_y = \frac{1}{3}\sigma_x$

Hence $\sigma_y = \frac{1}{3}\times 15 = 5$

**21. (a) :** Given that $\sum_{i=1}^{25}(x_i-40)^2 = 900$

$\Rightarrow \frac{1}{25}\sum_{i=1}^{25}(x_i-40)^2 = \frac{1}{25}\times 900 \Rightarrow \sigma_x^2 = 36 \Rightarrow \sigma_x = 6$

Therefore C.V. $= \frac{\sigma_x}{\bar{x}}\times 100 = \frac{6}{40}\times 100 = 15\%$

**22. (d) :** The sample space for the given experiment is
$S = \{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),(T,1),(T,2),$
$(T, 3), (T, 4), (T, 5), (T, 6)\}$

Let $A$ be the event of getting a head on the coin and an odd number on the die

$\therefore P(A) = \frac{3}{12} = \frac{1}{4}$

**23. (c) :** Clearly, $P(0 < x < 3) = P(x = 1) + P(x = 2)$

$$= \frac{{}^3C_1\times{}^7C_3}{{}^{10}C_4} + \frac{{}^3C_2\times{}^7C_2}{{}^{10}C_4} = \frac{4}{5}$$

**24. (d) :** Here the space consists of $7^5$ equally likely event points.

Let $A$ be the event that exactly 2 of 5 members have birthday on Sunday, then the number of favourable cases = ${}^5C_2 \times 6^3 = 10 \times 6^3$

Therefore, $P(A) = \frac{10\times 6^3}{7^5}$

**25. (b) :** Let $A$ be the event that the numbers formed by the digits 1,2,3,4,5 is divisible by 4.

Here, the number of exhaustive cases is 5! = 120.

Number is divisible by 4 when the last two digits of it are 12 or 24 or 32 or 52.

Thus in a 5-digits number the remaining 3 digits can arrange among themselves in 3! Ways.

Therefore number of favourable cases = 3! × 4 = 6 × 4 = 24

$\therefore P(A) = \frac{24}{120} = \frac{1}{5}$

**26. (d) :** Since in $xz$-plane, $y = 0$.

$\therefore$ The co-ordinates of the foot of the perpendicular drawn from the point $P(x, y, z)$ is $(x, 0, z)$.

**27. (d) :** The point of trisection of the line-segment joining the point (2, 1, –3) and (5, –8, 3) that is nearer to (5, –8, 3) is (4, –5, 1)

**28. (b) :** Given that $C$ is a point on the line segment $AB$, whose $y$-co-ordinate is –2.

Let $C$ divides the line segment $AB$ in the ratio $\lambda : 1$.

$\therefore -2 = \frac{\lambda\times 0 + 1(-3)}{\lambda+1} \Rightarrow -2(\lambda+1) = -3$

$\Rightarrow 2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$

Thus $C$ divides $AB$ in ratio 1 : 2.

$\therefore z = \frac{1\times 10 + 2\times 4}{1+2} = 6$

**29. (a) :** Let the line segment joining the points (3, 5, –7) and (–2, 1, 8) be divided by the $yz$-plane in the ratio $\lambda : 1$.

$\therefore \quad 0=\dfrac{\lambda\times(-2)+1\times 3}{\lambda+1} \Rightarrow -2\lambda+3=0 \Rightarrow \lambda=\dfrac{3}{2}$

Thus the required ratio is 3 : 2.

**30. (b) :** Let the vertex $A$ of the triangle $ABC$ be $(x, y, z)$.

Given that the centroid of triangle $ABC$ is $(3, -1, 3)$

$\therefore \quad 3=\dfrac{x+5+2}{3} \Rightarrow x=9-7=2,$

$-1=\dfrac{y+2-3}{3} \Rightarrow y=-2$ and

$3=\dfrac{z+8+4}{3} \Rightarrow z=-3$

$\therefore$ The co-ordinates of vertex $A$ are $(2, -2, -3)$.

**31. (b) :** The function $f(x)$ will be defined when

$\log_e(x^2-6x+6)\ge 0$

$\Rightarrow x^2-6x+6\ge e^0=1 \Rightarrow x^2-6x+5\ge 0$

$\Rightarrow (x-1)(x-5)\ge 0 \quad \ldots(1)$

Clearly the condition (1) will be satisfied when $x\le 1$ or $x\ge 5$.

Therefore the required domain is $(-\infty, 1]\cup[5,\infty)$

**32. (a) :** Let $M$ be the event that the selected person is a male and $S/M$ is the event that 'the selected person is a service holder' when it is known that he is a male.

Out of 10 men service holder, 1 may be selected in ${}^{10}C_1=10$ ways.

So, total number of cases favourable to the event $S/M$ is 10 and total number of exhaustive cases is 20.

Hence, the required probability $=P(S/M)=\dfrac{10}{20}=\dfrac{1}{2}.$

**33. (d) :** Let $A$'s selection be the event $E_1$ and that of $B$'s be $E_2$.

$\therefore \quad P(E_1)=\dfrac{1}{4},\ P(E_1^c)=\dfrac{3}{4},\ P(E_2)=\dfrac{1}{3},\ P(E_2^c)=\dfrac{2}{3}$

Required probability $=P[(E_1\cap E_2^c)\cup(E_2\cap E_1^c)]$

$=P(E_1\cap E_2^c)+P(E_2\cap E_1^c)$

$=P(E_1)P(E_2^c)+P(E_2)P(E_1^c)=\dfrac{1}{4}\cdot\dfrac{2}{3}+\dfrac{1}{3}\cdot\dfrac{3}{4}=\dfrac{5}{12}$

**34. (a) :** $\lim\limits_{x\to 4}\dfrac{x^{7/2}-4^{7/2}}{\log_e(x-3)}=\dfrac{\lim\limits_{x\to 4}\dfrac{x^{7/2}-4^{7/2}}{x-4}}{\lim\limits_{x\to 4}\dfrac{\log_e(x-3)}{x-4}}$

$=\dfrac{\dfrac{7}{2}\times 4^{5/2}}{\lim\limits_{h\to 0}\dfrac{\log_e(1+h)}{h}}=\dfrac{7}{2}\times 2^5=112$

**35. (c) :** The given function is

$f(x)=p[x+1]+q[x-1]$

Since, $\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^-} f(x)=f(1)$

Now, $\lim\limits_{x\to 1^+}\{p[x+1]+q[x-1]\}=\lim\limits_{x\to 1^-}\{p[x+1]+q[x-1]\}$

$\Rightarrow p\cdot 2+q\cdot 0=p\cdot 1+q\cdot(-1)$

$\Rightarrow 2p=p-q \Rightarrow p+q=0$

**36. (c) :** If $x$ is an integer, then

$[x+y]=x+[y]=x+[y+x-[x]]$, as $x-[x]=0$

If $n<x<n+1$, $x=n+k$, $0<k<1$.

Then $[x+y]=[n+k+y]=n+[y+k]$

$=[x]+[y+x-[x]]$, since $x-[x]=k$

**37. (a, d) :** $P(E/F)=\dfrac{P(E\cap F)}{P(F)},\ P(\bar{E}/F)=\dfrac{P(\bar{E}\cap F)}{P(F)}$

$\therefore \quad P(E/F)+P(\bar{E}/F)=\dfrac{P(E\cap F)+P(\bar{E}\cap F)}{P(F)}$

$=\dfrac{P\big((E\cap F)\cup(\bar{E}\cap F)\big)}{P(F)}=\dfrac{P(F)}{P(F)}=1$

[$\because$ $(E\cap F)$ and $(\bar{E}\cap F)$ are disjoint]

Similarly, $P(E/\bar{F})+P(\bar{E}/\bar{F})=\dfrac{P(\bar{F})}{P(\bar{F})}=1.$

**38. (a) :** $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$\Rightarrow \dfrac{5}{6}=\left(1-\dfrac{1}{2}\right)+P(B)-\dfrac{1}{3} \Rightarrow P(B)=\dfrac{2}{3}.$

$P(A)\cdot P(B)=\dfrac{1}{2}\cdot\dfrac{2}{3}=P(A\cap B).$

So, $A$ and $B$ are independent and therefore, not mutually exclusive.

**39. (b) :** As $\alpha$ is a repeated root, $a\alpha^2+b\alpha+c=0$ and $2a\alpha+b=0$

$\therefore \quad \lim\limits_{x\to\alpha}\dfrac{(2ax+b)\cos(ax^2+bx+c)}{2(x-\alpha)}$

$=\lim\limits_{x\to\alpha}\left[\dfrac{2a\cos(ax^2+bx+c)-(2ax+b)^2\sin(ax^2+bx+c)}{2}\right]$

$=\dfrac{2a}{2}=a$

**40. (a, d) :** $f(1^+)=\lim\limits_{h\to 0}\{|1+h-1|-[1+h]\}$

$=\lim\limits_{h\to 0}(h-1)=-1$

$f(1^-)=\lim\limits_{h\to 0}\{|1-h-1|-[1-h]\}=\lim\limits_{h\to 0}(h-0)=0.$

◈◈

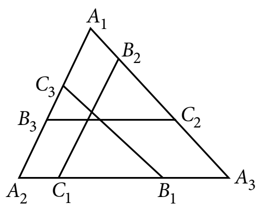
# OLYMPIAD CORNER

1. Ten players participate in a ping-pong tournament, in which every two players play against each other exactly once. If player $i$ beats player $j$, player $j$ beats player $k$ and player $k$ beats player $i$, then the set $\{i, j, k\}$ is called a triangle. Let $w_i$ and $l_i$ denote the number of games won and lost, respectively, by the $i^{th}$ player. Suppose that, whenever $i$ beats $j$, then $l_i + w_j \geq 8$. Prove that there are exactly 40 triangles in the tournament.

2. Triangle $ABC$ has incenter $I$ and centroid $G$. The line $IG$ intersects $BC$, $CA$, $AB$ in $K$, $L$, $M$ respectively. The line through $K$ parallel to $CA$ intersects the internal bisector of $\angle BAC$ in $P$. The line through $L$ parallel to $AB$ intersects the internal bisector of $\angle CBA$ in $Q$. The line through $M$ parallel to $BC$ intersects the internal bisector of $\angle ACB$ in $R$. Show that $BP$, $CQ$ and $AR$ are parallel.

3. In the figure, $B_2C_1 \,||\, A_1A_2$, $B_3C_2 \,||\, A_2A_3$ and $B_1C_3 \,||\, A_3A_1$. Prove that $B_2C_1$, $B_3C_2$ and $B_1C_3$ are concurrent if and only if

$$\frac{A_1C_3}{C_3B_3}\cdot\frac{A_2C_1}{C_1B_1}\cdot\frac{A_3C_2}{C_2B_2}=1.$$




4. Let $p$ be a prime. Find all solutions in positive integers of the equation:

$$\frac{2}{a}+\frac{3}{b}=\frac{5}{p}.$$

5. Find the value of the continued root:

$$\sqrt{4+27\sqrt{4+29\sqrt{4+31\sqrt{4+33\sqrt{\cdots}}}}}.$$

## SOLUTIONS

1. Let $P_i$ denote the $i^{th}$ player, $i = 1, 2, ..., 10$. Without loss of generality, assume that $w_1 \leq w_2 \leq ... \leq w_{10}$. Let $M = \max_{1 \leq i \leq 10} w_i = w_{10}$ and $m = \min_{1 \leq i \leq 10} w_i = w_1$. Since $w_i + l_i = 9$ for all $i$, the condition $l_i + w_j \geq 8$, whenever $P_i$ beats $P_j$, is equivalent to

$$w_i \leq w_j + 1 \qquad ...(1)$$

Clearly

$$\sum_{i=1}^{10} w_i = \binom{10}{2} = 45 \qquad ...(2)$$

We first show that

$$(w_1, w_2, ..., w_{10}) = (4,4,4,4,4,5,5,5,5,5)$$

Since $P_1$ beats $m$ players and loses to $9 - m$ others, we get from (1) and (2) that

$45 \leq m + Mm + (9 - m)(m + 1)$

or

$$36 \leq (M - m + 9)m \qquad ...(3)$$

Similarly, since $P_{10}$ beats $M$ players and loses to $9 - M$ others, we have

$$45 \geq M + M(M - 1) + (9 - M)m = M^2 + (9 - M)m \qquad ...(4)$$

Since $M \leq 9$, from (3) we obtain $36 \leq (18 - m)m$, which implies $m \geq 3$. From (4) we get $M^2 \leq 45$, *i.e.* $M \leq 6$. If $M = 6$, then from (4), $3m \leq 9$ or $m \leq 3$ and so $m = 3$. Similarly, if $m = 3$ then from (3) $M \geq 6$ and so $M = 6$. Since $10 \times 45$, $M \neq m$. Therefore, the only possibilities are $(M, m) = (6, 3)$ and $(5, 4)$.

Thus $(M, m) = (5, 4)$ and $(4, 4, 4, 4, 4, 5, 5, 5, 5, 5)$ is the only possible solution. By exhibiting the tournament graphically, one can see that this is indeed a solution.

Now, there is a theorem in graph theory which states that the number of transitive triples [3 players $P_i$, $P_j$ and $P_k$ such that $P_i$ beats $P_j$, $P_j$ beats $P_k$ and $P_i$

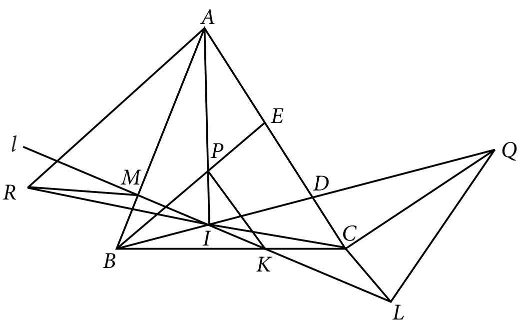
beats $P_k$] in a tournament with $n$ vertices and score sequence $(w_1, w_2, ...., w_n)$ is $\sum_{i=1}^{n} w_i(w_i - 1)/2$. It follows that, with $n = 10$, the number of triangles (cyclic triples) is

$$\binom{10}{3} - \frac{1}{2}\sum_{i=1}^{10} w_i(w_i - 1)$$
$$= 120 - \frac{1}{2}(5)(5)(5-1) - \frac{1}{2}(5)(4)(4-1) = 40.$$

This completes the proof.

**2.** We shall prove the following generalization:

*$ABC$ is a triangle and a line $l$ intersects $BC$, $CA$ and $AB$ at $K$, $L$ and $M$ respectively. $I$ is a point on $l$ other than $K$, $L$ or $M$. Let $P$, $Q$ and $R$ be points on $AI$, $BI$ and $CI$ respectively, such that $KP \mid\mid CA$, $LQ \mid\mid AB$ and $MR \mid\mid BC$. Then $BP$, $CQ$ and $AR$ are parallel.*




*Proof.* Let $BI$ and $BP$ intersect $AC$ at $D$ and $E$ respectively. Because $PK \mid\mid AC$ we get

$$\frac{EB}{BP} = \frac{CB}{BK} \text{ and } \frac{PI}{IA} = \frac{KI}{IL} \quad (1)$$

As $B$, $I$ and $D$ lie on a line, we have by Menelaus' theorem applied to $\Delta AEP$,

$$\frac{AD}{DE} \cdot \frac{EB}{BP} \cdot \frac{PI}{IA} = 1$$

From (1) this becomes

$$\frac{AD}{DE} \cdot \frac{CB}{BK} \cdot \frac{KI}{IL} = 1 \quad (2)$$

As $B$, $I$ and $D$ lie on a line, we have by Menelaus' theorem applied to $\Delta KCL$,

$$\frac{LD}{DC} \cdot \frac{CB}{BK} \cdot \frac{KI}{IL} = 1 \quad (3)$$

From (2) and (3) we have $AD/DE = LD/DC$, *i.e.*,

$$\frac{AD}{LD} = \frac{DE}{DC} \quad (4)$$

Because $AB \mid\mid LQ$ we get

$$\frac{AD}{LD} = \frac{BD}{QD} \quad (5)$$

From (4) and (5) we have

$$\frac{DE}{DC} = \frac{BD}{QD}$$

and therefore $BE \mid\mid CQ$, *i.e.* $BP \mid\mid CQ$. Similarly we can prove $AR \mid\mid CQ$.

**3.** Choose $A_1$ as origin and let $\overrightarrow{A_1A_2} = \vec{x}$ and $\overrightarrow{A_1A_3} = \vec{y}$. Suppose

$\overrightarrow{A_1C_3} = p\vec{x}$, $\overrightarrow{A_1B_2} = q\vec{y}$, where $p, q \neq 0$.

Then, since $C_3B_1$ is parallel to $A_1A_3$ and $B_1$ lies on $A_2A_3$, the position vector for $B_1$ must be $p\vec{x} + (1 - p)\vec{y}$. Similarly, the position vector for $C_1$ must be $q\vec{y} + (1 - q)\vec{x}$. Let the intersection of $C_3B_1$ and $B_2C_1$ be $P$; since $A_1C_3PB_2$ is a parallelogram, the position vector of $P$ is $p\vec{x} + q\vec{y}$. Since $B_3C_2$ is parallel to $A_2A_3$ it follows that if $B_3$ has position vector $u\vec{x}$ then $C_2$ has position vector $u\vec{y}$. Also $u \neq 1$ since $B_3C_2$ is distinct from $A_2A_3$.

The equation of $B_3C_2$ is $r = tu\vec{y} + (u - tu)\vec{x}$, $t \in R$, and this line passes through $P$ if and only if $u = p + q$ (with $t = q/(p + q)$). In other words, $B_1C_3$, $B_2C_1$, $B_3C_2$ are concurrent if and only if $u = p + q$.

In terms of $p$, $q$, $u$ we have

$$\frac{A_1C_3}{C_3B_3} \cdot \frac{A_2C_1}{C_1B_1} \cdot \frac{A_3C_2}{C_2B_2} = \frac{p}{u-p} \cdot \frac{q}{1-p-q} \cdot \frac{u-1}{q-u}.$$

This product equals 1 if and only if $pq(u - 1) = (u - p)(1 - p - q)(q - u)$, which holds (after some arithmetic) if and only if

$$u = p + q \quad \text{or} \quad u = \frac{pq}{p+q-1}$$

Thus the condition that the product be 1 is apparently necessary but not sufficient. (For example, let $p = q = -u = 1/3$.) If the given lines were required to intersect the interior of the triangle, then $u$ could not equal $pq/(p + q - 1)$ since the three intersection points are all inside the triangle if and only if $0 < p, q, p + q < 1$. We conclude therefore: if $B_1$, $B_2$, $B_3$ are points on the interior of the sides of $\Delta A_1A_2A_3$ then $B_1C_3$, $B_2C_1$, $B_3C_2$ are concurrent if and only if

$$\frac{A_1C_3}{C_3B_3}\cdot\frac{A_2C_1}{C_1B_1}\cdot\frac{A_3C_2}{C_2B_2}=1.$$

This is not entirely satisfactory because we would like a condition involving directed line segments that would be necessary and sufficient whether or not the given lines meet the interior of the triangle. With the usual sign conventions for directed line segments, one such condition is

$$\frac{A_1B_3}{A_1A_2}+\frac{A_2B_1}{A_2A_3}+\frac{A_3B_2}{A_3A_1}=2,$$

since this gives the equation $u + (1 - p) + (1 - q) = 2$, which holds if and only if $u = p + q$, which (as we saw) holds if and only if the given lines are concurrent.

**4.** Equivalently, we have $p(3a + 2b) = 5ab$; hence there are three cases to consider.

First case: $p = 5$. Then we get

$3a + 2b = ab$ or $(a - 2)(b - 3) = 6 = 1 \cdot 6 = 2 \cdot 3$.

Hence all pairs of positive integers $(a, b)$ are $(3, 9)$, $(4, 6)$, $(5, 5)$ and $(8, 4)$.

Second case: $p$ divides $a$. Let $a = a_1p$. Hence

$3a_1p = b(5a_1 - 2)$.

Then $p$ divides either $b$ or $5a_1 - 2$. If $b = b_1p$, then $3a_1 = b_1(5a_1 - 2)$ or $(5a_1 - 2)(5b_1 - 3) = 6$ which has only one solution: $(a_1, b_1) = (1, 1)$, whence $(a, b) = (p, p)$. Otherwise, $5a_1 - 2 = a_2p$. Therefore,

$$3\frac{a_2p+2}{5}=ba_2 \quad \text{or} \quad 3a_2p + 6 = 5ba_2.$$

Hence $a_2$ divides 6.

If $a_2 = 1$, then $(a, b) = \left(\frac{p(p+2)}{5},\frac{3(p+2)}{5}\right)$, but only if $p \equiv 3 \pmod 5$.

If $a_2 = 2$, then $(a, b) = \left(\frac{2p(p+1)}{5},\frac{3(p+1)}{5}\right)$, but only if $p \equiv 4 \pmod 5$.

If $a_2 = 3$, then $(a, b) = \left(\frac{p(3p+2)}{5},\frac{3p+2}{5}\right)$, but only if $p \equiv 1 \pmod 5$.

If $a_2 = 6$, then $(a, b) = \left(\frac{2p(3p+1)}{5},\frac{3p+1}{5}\right)$, but only if $p \equiv 3 \pmod 5$.

Third case: $p$ divides $b$. Let $b = b_1p$. Hence

$2b_1p = a(5b_1 - 3)$.

Then $p$ divides either $a$ or $5b_1 - 3$. If $p$ divides $a$, then we have the same case ($p$ divides $a$ and $b$) as already considered above. Again $(p, p)$ is the (only) solution. Otherwise, $5b_1 - 3 = b_2p$. Therefore,

$$2\frac{b_2p+3}{5}=ab_2 \quad \text{or} \quad 2b_2p + 6 = 5ab_2.$$

Hence $b_2$ divides 6.

If $b_2 = 1$, then $(a, b) = \left(\frac{2(p+3)}{5},\frac{p(p+3)}{5}\right)$, but only if $p \equiv 2 \pmod 5$.

If $b_2 = 2$, then $(a, b) = \left(\frac{2p+3}{5},\frac{p(2p+3)}{5}\right)$, but only if $p \equiv 1 \pmod 5$.

If $b_2 = 3$, then $(a, b) = \left(\frac{2(p+1)}{5},\frac{3p(p+1)}{5}\right)$, but only if $p \equiv 4 \pmod 5$.

If $b_2 = 6$, then $(a, b) = \left(\frac{2p+1}{5},\frac{3p(2p+1)}{5}\right)$, but only if $p \equiv 2 \pmod 5$.

[Note that if $p = 2$ or 3, this generates only two distinct solutions: $(2, 2)$, $(1, 6)$ for $p = 2$ and $(3, 3)$ $(12, 2)$ for $p = 3$. If $p > 5$, then the three solutions are all distinct.]

**5.** The answer is 29. More generally, for any positive integer $n$, we claim that

$$\sqrt{4+n\sqrt{4+(n+2)\sqrt{4+(n+4)\sqrt{\cdots}}}}=n+2,$$

where the left side is defined as the limit of

$$F(n, m)=\sqrt{4+n\sqrt{4+(n+2)\sqrt{4+(n+4)\sqrt{\ldots\sqrt{4+m\sqrt{4}}}}}}$$

as $m \to \infty$ (where $m$ is an integer and $(m - n)$ is even)

If $g(n, m) = F(n, m) - (n + 2)$, we have

$$F(n, m)^2-(n+2)^2=(4+nF(n+2,m))-(4+n(n+4))$$
$$= n(F(n + 2, m) - (n + 4)),$$

so $g(n, m)=\dfrac{n}{F(n,m)+n+2}g(n+2,m)$.

Clearly $F(n, m) > 2$, so

$$|g(n,m)|<\frac{n}{n+4}|g(n+2,m)|.$$

By iterating this, we obtain

$$|g(n,m)|<\frac{n(n+2)}{m(m+2)}|g(m,m)|<\frac{n(n+2)}{m}$$

Therefore $g(n, m) \to 0$ as $m \to \infty$.

*Contd. from Page No. 30*

**23. (b) :** Let the chord get bisected at $A(0, \alpha)$.
Centre of given circle is $C(1, 2)$.

$\therefore \quad m_{PA} \cdot m_{CA} = -1 \;\Rightarrow\; \dfrac{\alpha - b}{0 - a} \cdot \dfrac{\alpha - 2}{0 - 1} = -1$

$\Rightarrow \quad \alpha^2 - \alpha(b + 2) + a + 2b = 0$

This equation should have distinct real roots,

$\Rightarrow \quad (b + 2)^2 > 4\,(a + 2b) \;\Rightarrow\; b^2 - 4b + 4 > 4a$

**24. (d) :** Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$
We have, $2ga_1 + 2fb_1 = c + c_1$, $2ga_2 + 2f\,b_2 = c + c_2$

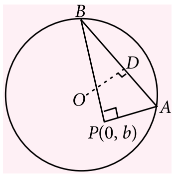
$\Rightarrow \quad 2g(a_1 - a_2) + 2f(b_1 - b_2) = c_1 - c_2$

Thus locus is

$$2x(a_1 - a_2) + 2y(b_1 - b_2) + c_1 - c_2 = 0$$

**25. (b) :** Clearly the points $A$, $B$ and $P$ are concyclic.

$\because \quad PD^2 = DA^2 = BD^2 = OA^2 - OD^2$




Let $D \equiv (x, y)$

$\Rightarrow \quad x^2 + (y - b)^2 = a^2 - (x^2 + y^2)$

$\Rightarrow \quad 2x^2 + 2y^2 - 2by + b^2 - a^2 = 0$

**26. (a) :** Equation of common tangent is

$$S_1 - S_2 = 0 \quad i.e.,\ x = y$$

Putting $x = y$ in the equation of circle, we get

$y^2 + y^2 + 2cy + b = 0 \;\Rightarrow\; 2y^2 + 2cy + b = 0$

It should have equal roots

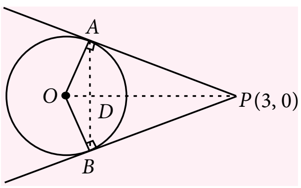
$(2c^2) - 4 \times 2 \times b = 0 \;\Rightarrow\; b = \dfrac{c^2}{2}$

Hence, $b > 0$

**27. (c) :** Equation of $AB$ is $T = 0$

*i.e.*, $3x + y \cdot 0 = 4$

*i.e.*, $x = 4/3 \Rightarrow OD = 4/3$




$\Rightarrow \quad AD^2 = OA^2 - OD^2$

$\Rightarrow \quad AD^2 = 4 - \dfrac{16}{9} = \dfrac{20}{9}$

$\Rightarrow \quad AD = \dfrac{2\sqrt{5}}{3}$

$\therefore \quad$ Area of $\Delta PAB = \dfrac{1}{2} \cdot \dfrac{4\sqrt{5}}{3} \cdot \left(3 - \dfrac{4}{3}\right) = \dfrac{10\sqrt{5}}{9}$ sq. units.

**28. (a) :** Any tangent of $x^2 + y^2 = 4$ is

$$y = mx \pm 2\sqrt{1 + m^2}$$

If it passes through $(-2, -4)$, then

$\Rightarrow \quad 4m^2 + 16 - 16m = 4 + 4m^2 \;\Rightarrow\; m = \infty, m = \dfrac{3}{4}$

Hence slope of reflected ray $= \dfrac{3}{4}$.

Thus equation of incident ray is

$(y + 4) = -\dfrac{3}{4}(x + 2)$ *i.e.*, $4y + 3x + 22 = 0$

**29. (b) :** Equation of radical axis of the given circles is $x = 0$.

If one circle lies completely inside the other, then centre of both circles should lie on the same side of radical axis and radical axis should not intersect the circles.

$\Rightarrow \quad (-a_1)(-a_2) > 0 \;\Rightarrow\; a_1a_2 > 0$

and $y^2 + c = 0$ should have imaginary roots

$\Rightarrow \quad c > 0$

**30. (c) :** Equation of any circles through $(0, 1)$ and $(0, 6)$ is $x^2 + (y - 1)(y - 6) + \lambda x = 0$

$\Rightarrow \quad x^2 + y^2 + \lambda x - 7y + 6 = 0$

If it touches $x$-axis then $x^2 + \lambda x + 6 = 0$ should have equal roots $\;\Rightarrow\; \lambda^2 = 24 \;\Rightarrow\; \lambda = \pm\sqrt{24}$

Radius of these circles $= \sqrt{6 + \dfrac{49}{4} - 6} = \dfrac{7}{2}$ units

*i.e.*, We can draw two circles but radius of both circles is 7/2.

**31. (a) :** Equation of any circles passing through $(1, 0)$ and $(5, 0)$ is $(x - 1)(x - 5) + y^2 + \lambda y = 0$

*i.e.*, $x^2 + y^2 + \lambda y - 6x + 5 = 0$

This circle touch the $y$-axis at $(0, h)$.

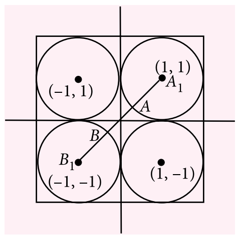
Putting $x = 0$ in the equation of circle, we get

$$y^2 + \lambda y + 5 = 0$$

It should have $y = h$ as it's repeated root

$\Rightarrow \quad h^2 = 5$ and $\lambda = -2h \;\Rightarrow\; |h| = \sqrt{5}$

**32. (d) :** $A_1B_1 = \sqrt{4 + 4} = 2\sqrt{2}$

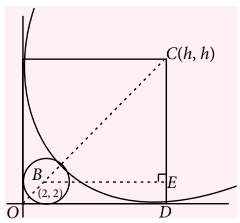



$\Rightarrow \quad AB = 2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$

Thus equation of required circle is

$x^2 + y^2 = (\sqrt{2} - 1)^2 = 3 - 2\sqrt{2}$

**33. (b) :** Let the centre of required circle be $(h, h)$.




We have, $\angle COD = \angle CBE = \pi/4$

$\Rightarrow \quad \dfrac{h - 2}{h + 2} = \cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad h = \dfrac{2(\sqrt{2} + 1)}{(\sqrt{2} - 1)} = 2(3 + 2\sqrt{2})$

**34. (d) :** $x^2 + 2ax + c = (x - 2)^2$

$\Rightarrow \quad -2a = 4,\ c = 4 \quad \Rightarrow\; a = -2,\ c = 4$

Also, $y^2 + 2by + c = (y - 2)(y - 3)$

$\Rightarrow \quad -2b = 5,\ c = 6 \;\Rightarrow\; b = -\dfrac{5}{2},\ c = 6$

Clearly, the data is not consistent.

**35. (c)**

# YOU ASK WE ANSWER

**Do you have a question that you just can't get answered?**

Use the vast expertise of our MTG team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough.

The best questions and their solutions will be printed in this column each month.

**1.** Evaluate: $\lim\limits_{x\to0^+}\left(\lim\limits_{n\to\infty}\dfrac{[1^2x^x]+[2^2x^x]+.....+[n^2x^x]}{n^3}\right)$

where $[\cdot]$ denotes integral part. ***Akash, Delhi***

**Ans.** We have, $1^2x^x-1\le[1^2x^x]\le1^2x^x$

$2^2x^x-1\le[2^2x^x]\le2^2x^x$

............

............

............

$n^2x^x-1\le[n^2x^x]\le n^2x^x$

Adding the above inequations,

$$\frac{x^x\Sigma n^2-n}{n^3}\le\frac{\Sigma[n^2x^x]}{n^3}\le\frac{x^x\Sigma n^2}{n^3}$$

$$i.e.,\ x^x\frac{n(n+1)(2n+1)}{6n^3}-\frac{1}{n^2}\le\frac{\Sigma[n^2x^x]}{n^3}\le x^x\frac{n(n+1)(2n+1)}{6n^3}$$

Now, applying $\lim\limits_{n\to\infty}$, we have

$$\frac{x^x}{3}\le\lim_{n\to\infty}\frac{\Sigma[n^2x^x]}{n^3}\le\frac{x^x}{3}$$

Hence, by Sandwich Theorem, we have

$$\lim_{n\to\infty}\frac{\Sigma[n^2x^x]}{n^3}=\frac{x^x}{3}$$

Now, the required limit, is

$$\lim_{x\to0^+}\left(\lim_{n\to\infty}\frac{\Sigma[n^2x^x]}{n^3}\right)=\frac{1}{3}\lim_{x\to0^+}x^x=\frac{1}{3}\lim_{x\to0^+}e^{\frac{\ln x}{1/x}}$$

$$=\frac{1}{3}\lim_{x\to0^+}e^{\frac{1/x}{-1/x^2}}=\frac{1}{3}e^0=\frac{1}{3}.$$

**2.** $A$ tosses 2 fair coins and $B$ tosses 3 fair coins. The game is won by the person who throws greater number of heads. In case of a tie, the game is continued under identical rules until someone finally wins the game. Find the probability that $A$ finally wins the game. ***Himanshu, U.P.***

**Ans.** For a particular game, let

$A_i$ = number of heads obtained by $A$ when he tosses two coins and $B_i$ = number of heads obtained by $B$ when he tosses three coins. If $E$ = {$A$ wins a particular game}, then

$P(E)=P(A_1\cap B_0\ \text{or}\ A_2\cap B_0\ \text{or}\ A_2\cap B_1)$

$=P(A_1\cap B_0)+P(A_2\cap B_0)+P(A_2\cap B_1)$

$=P(A_1)\,P(B_0)+P(A_2)\,P(B_0)+P(A_2)\,P(B_1)$

$$={}^2C_1\left(\frac12\right)\left(\frac12\right)\cdot\left(\frac12\right)^3+\left(\frac12\right)^2\left(\frac12\right)^3+\left(\frac12\right)^2\cdot{}^3C_1\left(\frac12\right)\left(\frac12\right)^2$$

$$=\frac{2}{32}+\frac{1}{32}+\frac{3}{32}=\frac{3}{16}$$

If $F$ = {$A$ and $B$ tie a particular game}, then

$P(F)=P(A_0\cap B_0\ \text{or}\ A_1\cap B_1\ \text{or}\ A_2\cap B_2)$

$=P(A_0)\,P(B_0)+P(A_1)\,P(B_1)+P(A_2)\,P(B_2)$

$$=\left(\frac12\right)^2\left(\frac12\right)^3+{}^2C_1\left(\frac12\right)\left(\frac12\right)\cdot{}^3C_1\left(\frac12\right)\left(\frac12\right)^2+\left(\frac12\right)^2\cdot{}^3C_2\left(\frac12\right)^2\left(\frac12\right)=\frac{5}{16}$$

Hence, $P(A$ finally wins the game)

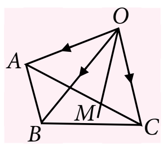
$=P(E\ \text{or}\ FE\ \text{or}\ FFE\ \text{or}\ .........)$

$=P(E)+P(F)\,P(E)+\{P(F)\}^2\,P(E)+........$

$$=\frac{P(E)}{1-P(F)}=\frac{3/16}{1-5/16}=\frac{3}{11}.$$

**3.** If $l$ is the length of an edge of a regular tetrahedron, then find the distance of any vertex from its opposite face. ***Sakshi, Kerala***

**Ans.** The figure shows a regular tetrahedron $OABC$. Let $O$ be the origin and the position vectors of the vertices $A$, $B$, $C$ be $a$, $b$, $c$ respectively. Then, we have




Volume of $OABC=\dfrac13\,ar\,(\Delta\,ABC)\,(OM)$

$i.e.,\ \dfrac16[a\ b\ c]=\dfrac13\,ar\,(\Delta\,ABC)\,(OM)$

Now, we have

$ar(\Delta\,ABC)=\dfrac{\sqrt3}{4}l^2$ $\qquad[\because\ \Delta\,ABC$ is equilateral]

$$\text{and }[a\,b\,c]^2=\begin{vmatrix}a\cdot a & a\cdot b & a\cdot c\\ b\cdot a & b\cdot b & b\cdot c\\ c\cdot a & c\cdot b & c\cdot c\end{vmatrix}$$

$$=l^6\begin{vmatrix}1 & \cos60^\circ & \cos60^\circ\\ \cos60^\circ & 1 & \cos60^\circ\\ \cos60^\circ & \cos60^\circ & 1\end{vmatrix}=\frac12 l^6$$

$$\text{Hence, }\frac16\frac{l^3}{\sqrt2}=\frac13\left(\frac{\sqrt3}{4}l^2\right)(OM)\ \ i.e.,\ OM=\sqrt{\frac23}\,l.$$

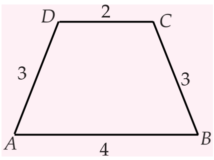
◈◈

# MATHS MUSING

## SOLUTION SET-167

1. **(d):** Let $A\left(ct_1, \frac{c}{t_1}\right), B\left(ct_2, \frac{c}{t_2}\right), C\left(ct_3, \frac{c}{t_3}\right)$ be the vertices of triangle lying on the curve $xy = c^2$. The orthocentre is $\left(\frac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right)$, which lies on the curve.

2. **(b):** $A = (-2, 0), B = (2, 0)$




$C = (1, 2\sqrt{2}),\ D = (-1, 2\sqrt{2})$

If the desired circle has centre $(0, r)$ and radius $r$, then

$4 + r^2 = (r + 2)^2,\ 1 + (r - 2\sqrt{2})^2 = (r + 1)^2$

Subtracting, $r = \frac{r+4}{2\sqrt{2}}$

$\therefore\ \frac{(r+4)^2}{8} = r^2 + 4r \Rightarrow 7r^2 + 24r - 16 = 0 \Rightarrow r = \frac{4}{7}.$

3. **(c):** If $\alpha_1, \alpha_2, \ldots\ldots, \alpha_6$ be the roots of $f(x) = 0$, then

$\sum_{r=1}^{6} \alpha_r = 12, \alpha_1\alpha_2 \ldots\ldots \alpha_6 = 64 \quad \therefore$ A.M. = G.M. = 2

The roots are equal and $f(x) = (x - 2)^6$

$\therefore\ f''(1) = 6 \times 5 = 30$

4. **(c):** $f(x) = \int_0^x e^{x-t}dt + \int_x^4 e^{t-x}dt = e^x + e^{4-x} - 2$

$b = f(0) = f(4) = e^4 - 1$ and $a = f(2) = 2e^2 - 2$

$\Rightarrow\ b - a = e^4 - 2e^2 + 1,\ \sqrt{b-a} = e^2 - 1.$

5. **(a):** $\frac{x^2}{2} + \frac{y^2}{1} = 1 \Rightarrow P \equiv \left(-1, \frac{1}{\sqrt{2}}\right)$ and $Q \equiv \left(1, \frac{1}{\sqrt{2}}\right)$.

If $PQ$ is the latus rectum of the parabola then $PQ = 2$.

Vertex is $\left(0, \frac{1}{\sqrt{2}} + \frac{1}{2}\right), \left(0, \frac{1}{\sqrt{2}} - \frac{1}{2}\right)$

The parabolas are

$x^2 = -2\left(y - \frac{1}{\sqrt{2}} - \frac{1}{2}\right)$ *i.e.,* $x^2 + 2y = \sqrt{2} + 1$ and

$x^2 = 2\left(y - \frac{1}{\sqrt{2}} + \frac{1}{2}\right)$ *i.e.,* $x^2 - 2y = 1 - \sqrt{2}.$

6. **(a, b, c):** $S = \sum_{r=0}^{5} (z_1 + z_2\alpha^r)(\bar{z}_1 + \bar{z}_2\alpha^{\bar{r}})$

$= \sum_{r=0}^{5} (z_1 + z_2\alpha^r)(\bar{z}_1 + \bar{z}_2\alpha^{6-r})$

$= \sum_{r=0}^{5} \left[|z_1|^2 + |z_2|^2 + \bar{z}_1z_2\alpha^r + z_1\bar{z}_2\alpha^{6-r}\right]$

$= \sum_{r=0}^{5} \left(|z_1|^2 + |z_2|^2\right) \quad \left[\because\ \sum_{r=0}^{5} \alpha^r = 0\right]$

$= 6(25 + 5) = 180 = 2^2 \cdot 3^2 \cdot 5$

7. **(d):** $g(x) = \frac{x^2+x}{x-1},\ g'(x) = 0 \Rightarrow x = 1 \pm \sqrt{2}$

Min $g(x) = g(1 + \sqrt{2}) = 3 + 2\sqrt{2}$

Max $g(x) = g(1 - \sqrt{2}) = 3 - 2\sqrt{2}$

8. **(b):** Area $= \int_{-1}^{0} g(x)dx = \int_{-1}^{0} \frac{(x^2 - 1 + x - 1 + 2)}{x-1}dx$

$= \int_{-1}^{0} (x+2)dx + 2\ln|x-1|\Big|_{-1}^{0} = \frac{3}{2} - 2\ln 2 = \frac{3}{2} - \ln 4$

9. **(6):** $P\left(1, \frac{1}{\sqrt{2}}\right), S(1, 0), S'(-1, 0)$

$PS$ meets the curve $x^2 + 2y^2 = 2$ at $Q\left(1, -\frac{1}{\sqrt{2}}\right)$

$PS'$ meets it at $R\left(-\frac{7}{5}, -\frac{1}{5\sqrt{2}}\right)$

$QR^2 = \left(1 + \frac{7}{5}\right)^2 + \left(\frac{1}{5\sqrt{2}} - \frac{1}{\sqrt{2}}\right)^2 = \frac{152}{25}$

$d^2 = 6.08 \Rightarrow [d^2] = 6$

10. **(a) → (q), (b) → (t), (c) → (p), (d) → (q)**

(a) $\lim_{x\to\infty} \frac{\int_0^x e^{t^2}dt}{e^{x^2}} = \lim_{x\to\infty} \frac{1}{2 - \frac{1}{x^2}} = \frac{1}{2}$ [by using L.H. rule]

(b) $f(x) = x^2 + \int_0^x e^{-t} f(x-t)dt$

$= x^2 + \int_0^x f(t)e^{t-x} \cdot dt = x^2 + e^{-x}\int_0^x e^t f(t)dt$

$\therefore\ f'(x) = 2x + f(x) - e^{-x} \cdot \int_0^x e^t f(t)dt$

$\Rightarrow\ f'(x) = x^2 + 2x \Rightarrow f(x) = \frac{x^3}{3} + x^2 \Rightarrow f(1) = \frac{4}{3}$

(c) $f(x) = \int_0^x (x-t)dt + \int_x^1 (t-x)dt = \frac{1}{2} - x + x^2$

$\therefore\ \int_0^1 f(x)dx = \frac{1}{3}$

(d) Let $I = \int_0^{\pi/2} \frac{1 + 2\cos x}{(2 + \cos x)^2}dx \quad \left(\text{Put } t = \tan\frac{x}{2}\right)$

$\therefore\ I = 2\int_0^1 \frac{3 - t^2}{(3 + t^2)^2}dt \quad (\text{Put } t = \sqrt{3}\tan\theta)$

$\Rightarrow\ I = \frac{2}{\sqrt{3}}\int_0^{\pi/6} \cos 2\theta\, d\theta = \frac{1}{2}$

◇◇