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# ELECTRODYNAMICS OF CONTINUOUS MEDIA

by

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*Institute of Physical Problems, USSR Academy of Sciences*

*Volume 8 of Course of Theoretical Physics*

*Translated from the Russian by*

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## PREFACE TO THE SECOND EDITION

TWENTY-FIVE years have passed since the writing of this volume in its first edition. Such a long interval has inevitably made necessary a fairly thorough revision and expansion of the book for its second edition.

The original choice of material was such that, with some very slight exceptions, it has not become obsolete. In this part, only some relatively minor additions and improvements have been made.

It has, however, been necessary to incorporate a considerable amount of new material. This relates in particular to the theory of the magnetic properties of matter and the theory of optical phenomena, with new chapters on spatial dispersion and non-linear optics.

The chapter on electromagnetic fluctuations has been deleted, since this topic is now dealt with, in a different way, in Volume 9 of the *Course*.

As with the other volumes, invaluable help in the revision has been derived from the comments of scientific colleagues, who are too numerous to be named here in their entirety, but to whom we offer our sincere thanks. Particularly many comments came from V. L. Ginzburg, B. Ya. Zel'dovich and V. P. Kraĭnov. It was most useful to be able to hold regular discussions of questions arising, with A. F. Andreev, I. E. Dzyaloshinskiĭ and I. M. Lifshitz. We are particularly grateful to S. I. Vaĭnshteĭn and R. V. Polovin for much assistance in revising the chapter on magnetohydrodynamics. Lastly, our thanks are due to A. S. Borovik-Romanov, V. I. Grigor'ev and M. I. Kaganov for reading the manuscript and for a number of useful remarks.

Moscow  
July, 1981

E. M. LIFSHITZ  
L. P. PITAEVSKIĬ



## PREFACE TO THE FIRST ENGLISH EDITION

THE present volume in the *Course of Theoretical Physics* deals with the theory of electromagnetic fields in matter and with the theory of the macroscopic electric and magnetic properties of matter. These theories include a very wide range of topics, as may be seen from the Contents.

In writing this book we have experienced considerable difficulties, partly because of the need to make a selection from the extensive existing material, and partly because the customary exposition of many topics to be included does not possess the necessary physical clarity, and sometimes is actually wrong. We realize that our own treatment still has many defects, which we hope to correct in future editions.

We are grateful to Professor V. L. Ginzburg, who read the book in manuscript and made some useful comments. I. E. Dzyaloshinskii and L. P. Pitaevskii gave great help in reading the proofs of the Russian edition. Thanks are due also to Dr Sykes and Dr Bell, who not only carried out excellently the arduous task of translating the book, but also made some useful comments concerning its contents.

Moscow  
June, 1959

L. D. LANDAU  
E. M. LIFSHITZ

## NOTATION

Electric field  $\mathbf{E}$

Electric induction  $\mathbf{D}$

Magnetic field  $\mathbf{H}$

Magnetic induction  $\mathbf{B}$

External electric field  $\mathcal{E}$ , magnitude  $\mathcal{E}$

External magnetic field  $\mathcal{H}$ , magnitude  $\mathcal{H}$

Dielectric polarization  $\mathbf{P}$

Magnetization  $\mathbf{M}$

Total electric moment of a body  $\mathcal{P}$

Total magnetic moment of a body  $\mathcal{M}$

Permittivity  $\epsilon$

Dielectric susceptibility  $\kappa$

Magnetic permeability  $\mu$

Magnetic susceptibility  $\chi$

Current density  $\mathbf{j}$

Conductivity  $\sigma$

Absolute temperature (in energy units)  $T$

Pressure  $P$

Volume  $V$

|                           |                 |            |
|---------------------------|-----------------|------------|
| Thermodynamic quantities: | per unit volume | for a body |
|---------------------------|-----------------|------------|

|         |     |               |
|---------|-----|---------------|
| entropy | $S$ | $\mathcal{S}$ |
|---------|-----|---------------|

|                 |     |               |
|-----------------|-----|---------------|
| internal energy | $U$ | $\mathcal{U}$ |
|-----------------|-----|---------------|

|             |     |               |
|-------------|-----|---------------|
| free energy | $F$ | $\mathcal{F}$ |
|-------------|-----|---------------|

|  |        |               |
|--|--------|---------------|
| thermodynamic potential<br>(Gibbs free energy) | $\Phi$ | $\mathcal{G}$ |
|--|--------|---------------|

Chemical potential  $\zeta$

A complex periodic time factor is always taken as  $e^{-i\omega t}$ .

Volume element  $dV$  or  $d^3x$ ; surface element  $df$ .

The summation convention always applies to three-dimensional (Latin) and two-dimensional (Greek) suffixes occurring twice in vector and tensor expressions.

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*QED* = Vol. 4 (*Quantum Electrodynamics*, second English edition, 1982).

*SP 1* = Vol. 5 (*Statistical Physics*, Part 1, third English edition, 1980).

*FM* = Vol. 6 (*Fluid Mechanics*, English edition, 1959).

*TE* = Vol. 7 (*Theory of Elasticity*, second English edition, 1970).

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## CHAPTER I

# ELECTROSTATICS OF CONDUCTORS

### §1. The electrostatic field of conductors

MACROSCOPIC electrodynamics is concerned with the study of electromagnetic fields in space that is occupied by matter. Like all macroscopic theories, electrodynamics deals with physical quantities averaged over elements of volume which are “physically infinitesimal”, ignoring the microscopic variations of the quantities which result from the molecular structure of matter. For example, instead of the actual “microscopic” value of the electric field  $\mathbf{e}$ , we discuss its averaged value, denoted by  $\mathbf{E}$ :

$$\bar{\mathbf{e}} = \mathbf{E}. \quad (1.1)$$

The fundamental equations of the electrodynamics of continuous media are obtained by averaging the equations for the electromagnetic field in a vacuum. This method of obtaining the macroscopic equations from the microscopic was first used by H. A. Lorentz (1902).

The form of the equations of macroscopic electrodynamics and the significance of the quantities appearing in them depend essentially on the physical nature of the medium, and on the way in which the field varies with time. It is therefore reasonable to derive and investigate these equations separately for each type of physical object.

It is well known that all bodies can be divided, as regards their electric properties, into two classes, *conductors* and *dielectrics*, differing in that any electric field causes in a conductor, but not in a dielectric, the motion of charges, i.e. an *electric current*.†

Let us begin by studying the static electric fields produced by charged conductors, that is, the *electrostatics of conductors*. First of all, it follows from the fundamental property of conductors that, in the electrostatic case, the electric field inside a conductor must be zero. For a field  $\mathbf{E}$  which was not zero would cause a current; the propagation of a current in a conductor involves a dissipation of energy, and hence cannot occur in a stationary state (with no external sources of energy).

Hence it follows, in turn, that any charges in a conductor must be located on its surface. The presence of charges inside a conductor would necessarily cause an electric field in it;‡ they can be distributed on its surface, however, in such a way that the fields which they produce in its interior are mutually balanced.

Thus the problem of the electrostatics of conductors amounts to determining the electric field in the vacuum outside the conductors and the distribution of charges on their surfaces.

At any point far from the surface of the body, the mean field  $\mathbf{E}$  in the vacuum is almost

† The conductor is here assumed to be homogeneous (in composition, temperature, etc.). In an inhomogeneous conductor, as we shall see later, there may be fields which cause no motion of charges.

‡ This is clearly seen from equation (1.8) below.

## CHAPTER I

# ELECTROSTATICS OF CONDUCTORS

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‡ This is clearly seen from equation (1.8) below.

the same as the actual field  $\mathbf{e}$ . The two fields differ only in the immediate neighbourhood of the body, where the effect of the irregular molecular fields is noticeable, and this difference does not affect the averaged field equations. The exact microscopic Maxwell's equations in the vacuum are

$$\text{div } \mathbf{e} = 0. \quad (1.2)$$

$$\text{curl } \mathbf{e} = -(1/c)\partial \mathbf{h}/\partial t, \quad (1.3)$$

where  $\mathbf{h}$  is the microscopic magnetic field. Since the mean magnetic field is assumed to be zero, the derivative  $\partial \mathbf{h}/\partial t$  also vanishes on averaging, and we find that the static electric field in the vacuum satisfies the usual equations

$$\text{div } \mathbf{E} = 0, \quad \text{curl } \mathbf{E} = 0, \quad (1.4)$$

i.e. it is a potential field with a potential  $\phi$  such that

$$\mathbf{E} = -\text{grad } \phi, \quad (1.5)$$

and  $\phi$  satisfies Laplace's equation

$$\Delta \phi = 0. \quad (1.6)$$

The boundary conditions on the field  $\mathbf{E}$  at the surface of a conductor follow from the equation  $\text{curl } \mathbf{E} = 0$ , which, like the original equation (1.3), is valid both outside and inside the body. Let us take the  $z$ -axis in the direction of the normal  $\mathbf{n}$  to the surface at some point on the conductor. The component  $E_z$  of the field takes very large values in the immediate neighbourhood of the surface (because there is a finite potential difference over a very small distance). This large field pertains to the surface itself and depends on the physical properties of the surface, but is not involved in our electrostatic problem, because it falls off over distances comparable with the distances between atoms. It is important to note, however, that, if the surface is homogeneous, the derivatives  $\partial E_z/\partial x$ ,  $\partial E_z/\partial y$  along the surface remain finite, even though  $E_z$  itself becomes very large. Hence, since  $(\text{curl } \mathbf{E})_x = \partial E_z/\partial y - \partial E_y/\partial z = 0$ , we find that  $\partial E_y/\partial z$  is finite. This means that  $E_y$  is continuous at the surface, since a discontinuity in  $E_y$  would mean an infinity of the derivative  $\partial E_y/\partial z$ . The same applies to  $E_x$ , and since  $\mathbf{E} = 0$  inside the conductor, we reach the conclusion that the tangential components of the external field at the surface must be zero:

$$E_t = 0. \quad (1.7)$$

Thus the electrostatic field must be normal to the surface of the conductor at every point. Since  $\mathbf{E} = -\text{grad } \phi$ , this means that the field potential must be constant on the surface of any particular conductor. In other words, the surface of a homogeneous conductor is an equipotential surface of the electrostatic field.

The component of the field normal to the surface is very simply related to the charge density on the surface. The relation is obtained from the general electrostatic equation  $\text{div } \mathbf{e} = 4\pi\rho$ , which on averaging gives

$$\text{div } \mathbf{E} = 4\pi\bar{\rho}, \quad (1.8)$$

$\bar{\rho}$  being the mean charge density. The meaning of the integrated form of this equation is well known: the flux of the electric field through a closed surface is equal to the total charge inside that surface, multiplied by  $4\pi$ . Applying this theorem to a volume element lying between two infinitesimally close unit areas, one on each side of the surface of the

conductor, and using the fact that  $\mathbf{E} = 0$  on the inner area, we find that  $E_n = 4\pi\sigma$ , where  $\sigma$  is the surface charge density, i.e. the charge per unit area of the surface of the conductor. Thus the distribution of charges over the surface of the conductor is given by the formula

$$4\pi\sigma = E_n = -\partial\phi/\partial n, \quad (1.9)$$

the derivative of the potential being taken along the outward normal to the surface. The total charge on the conductor is

$$e = -\frac{1}{4\pi} \oint \frac{\partial\phi}{\partial n} df, \quad (1.10)$$

the integral being taken over the whole surface.

The potential distribution in the electrostatic field has the following remarkable property: the function  $\phi(x, y, z)$  can take maximum and minimum values only at boundaries of regions where there is a field. This theorem can also be formulated thus: a test charge  $e$  introduced into the field cannot be in stable equilibrium, since there is no point at which its potential energy  $e\phi$  would have a minimum.

The proof of the theorem is very simple. Let us suppose, for example, that the potential has a maximum at some point  $A$  not on the boundary of a region where there is a field. Then the point  $A$  can be surrounded by a small closed surface on which the normal derivative  $\partial\phi/\partial n < 0$  everywhere. Consequently, the integral over this surface  $\oint (\partial\phi/\partial n) df < 0$ . But by Laplace's equation  $\oint (\partial\phi/\partial n) df = \int \Delta\phi dV = 0$ , giving a contradiction.

## §2. The energy of the electrostatic field of conductors

Let us calculate the total energy  $\mathcal{U}$  of the electrostatic field of charged conductors,†

$$\mathcal{U} = \frac{1}{8\pi} \int \mathbf{E}^2 dV, \quad (2.1)$$

where the integral is taken over all space outside the conductors. We transform this integral as follows:

$$\mathcal{U} = -\frac{1}{8\pi} \int \mathbf{E} \cdot \text{grad } \phi dV = -\frac{1}{8\pi} \int \text{div}(\phi \mathbf{E}) dV + \frac{1}{8\pi} \int \phi \text{div } \mathbf{E} dV.$$

The second integral vanishes by (1.4), and the first can be transformed into integrals over the surfaces of the conductors which bound the field and an integral over an infinitely remote surface. The latter vanishes, because the field diminishes sufficiently rapidly at infinity (the arbitrary constant in  $\phi$  is assumed to be chosen so that  $\phi = 0$  at infinity). Denoting by  $\phi_a$  the constant value of the potential on the  $a$ th conductor, we have‡

$$\mathcal{U} = \frac{1}{8\pi} \sum_a \oint \phi E_n df = \frac{1}{8\pi} \sum_a \phi_a \oint E_n df.$$

† The square  $\mathbf{E}^2$  is not the same as the mean square  $\overline{\mathbf{e}^2}$  of the actual field near the surface of a conductor or inside it (where  $\mathbf{E} = 0$  but, of course,  $\mathbf{e}^2 \neq 0$ ). By calculating the integral (2.1) we ignore the internal energy of the conductor as such, which is here of no interest, and the affinity of the charges for the surface.

‡ In transforming volume integrals into surface integrals, both here and later, it must be borne in mind that  $E_n$  is the component of the field along the outward normal to the conductor. This direction is opposite to that of the outward normal to the region of the volume integration, namely the space outside the conductors. The sign of the integral is therefore changed in the transformation.

Finally, since the total charges  $e_a$  on the conductors are given by (1.10) we obtain

$$\mathcal{U} = \frac{1}{2} \sum_a e_a \phi_a, \quad (2.2)$$

which is analogous to the expression for the energy of a system of point charges.

The charges and potentials of the conductors cannot both be arbitrarily prescribed; there are certain relations between them. Since the field equations in a vacuum are linear and homogeneous, these relations must also be linear, i.e. they must be given by equations of the form

$$e_a = \sum_b C_{ab} \phi_b, \quad (2.3)$$

where the quantities  $C_{aa}, C_{ab}$  have the dimensions of length and depend on the shape and relative position of the conductors. The quantities  $C_{aa}$  are called *coefficients of capacity*, and the quantities  $C_{ab} (a \neq b)$  are called *coefficients of electrostatic induction*. In particular, if there is only one conductor, we have  $e = C\phi$ , where  $C$  is the *capacitance*, which in order of magnitude is equal to the linear dimension of the body. The converse relations, giving the potentials in terms of the charges, are

$$\phi_a = \sum_b C^{-1}_{ab} e_b, \quad (2.4)$$

where the coefficients  $C^{-1}_{ab}$  form a matrix which is the inverse of the matrix  $C_{ab}$ .

Let us calculate the change in the energy of a system of conductors caused by an infinitesimal change in their charges or potentials. Varying the original expression (2.1), we have  $\delta \mathcal{U} = (1/4\pi) \int \mathbf{E} \cdot \delta \mathbf{E} dV$ . This can be further transformed by two equivalent methods. Putting  $\mathbf{E} = -\mathbf{grad} \phi$  and using the fact that the varied field, like the original field, satisfies equations (1.4) (so that  $\text{div} \delta \mathbf{E} = 0$ ), we can write

$$\begin{aligned} \delta \mathcal{U} &= -\frac{1}{4\pi} \int \mathbf{grad} \phi \cdot \delta \mathbf{E} dV = -\frac{1}{4\pi} \int \text{div} (\phi \delta \mathbf{E}) dV \\ &= \frac{1}{4\pi} \sum_a \phi_a \oint \delta E_n df, \end{aligned}$$

that is

$$\delta \mathcal{U} = \sum_a \phi_a \delta e_a, \quad (2.5)$$

which gives the change in energy due to a change in the charges. This result is obvious; it is the work required to bring infinitesimal charges  $\delta e_a$  to the various conductors from infinity, where the field potential is zero.

On the other hand, we can write

$$\begin{aligned} \delta \mathcal{U} &= -\frac{1}{4\pi} \int \mathbf{E} \cdot \mathbf{grad} \delta \phi dV = -\frac{1}{4\pi} \int \text{div} (\mathbf{E} \delta \phi) dV \\ &= \frac{1}{4\pi} \sum_a \delta \phi_a \oint E_n df, \end{aligned}$$

that is

$$\delta \mathcal{U} = \sum_a e_a \delta \phi_a, \quad (2.6)$$

which expresses the change in energy in terms of the change in the potentials of the conductors.

Formulae (2.5) and (2.6) show that, by differentiating the energy  $\mathcal{U}$  with respect to the charges, we obtain the potentials of the conductors, and the derivatives of  $\mathcal{U}$  with respect to the potentials are the charges:

$$\partial \mathcal{U} / \partial e_a = \phi_a, \quad \partial \mathcal{U} / \partial \phi_a = e_a. \quad (2.7)$$

But the potentials and charges are linear functions of each other. Using (2.3) we have  $\partial^2 \mathcal{U} / \partial \phi_a \partial \phi_b = \partial e_b / \partial \phi_a = C_{ba}$ , and by reversing the order of differentiation we get  $C_{ab}$ . Hence it follows that

$$C_{ab} = C_{ba}, \quad (2.8)$$

and similarly  $C^{-1}_{ab} = C^{-1}_{ba}$ . The energy  $\mathcal{U}$  can be written as a quadratic form in either the potentials or the charges:

$$\mathcal{U} = \frac{1}{2} \sum_{a,b} C_{ab} \phi_a \phi_b = \frac{1}{2} \sum_{a,b} C^{-1}_{ab} e_a e_b. \quad (2.9)$$

This quadratic form must be positive definite, like the original expression (2.1). From this condition we can derive various inequalities which the coefficients  $C_{ab}$  must satisfy. In particular, all the coefficients of capacity are positive:

$$C_{aa} > 0 \quad (2.10)$$

(and also  $C^{-1}_{aa} > 0$ ).†

All the coefficients of electrostatic induction, on the other hand, are negative:

$$C_{ab} < 0 \quad (a \neq b). \quad (2.11)$$

That this must be so is seen from the following simple arguments. Let us suppose that every conductor except the  $a$ th is earthed, i.e. their potentials are zero. Then the charge induced by the charged  $a$ th conductor on another (the  $b$ th, say) is  $e_b = C_{ba} \phi_a$ . It is obvious that the sign of the induced charge must be opposite to that of the inducing potential, and therefore  $C_{ab} < 0$ . This can be more rigorously shown from the fact that the potential of the electrostatic field cannot reach a maximum or minimum outside the conductors. For example, let the potential  $\phi_a$  of the only conductor not earthed be positive. Then the potential is positive in all space, its least value (zero) being attained only on the earthed conductors. Hence it follows that the normal derivative  $\partial \phi / \partial n$  of the potential on the surfaces of these conductors is positive, and their charges are therefore negative, by (1.10). Similar arguments show that  $C^{-1}_{ab} > 0$ .

The energy of the electrostatic field of conductors has a certain extremum property, though this property is more formal than physical. To derive it, let us suppose that the

† We may also mention that another inequality which must be satisfied if the form (2.9) is positive is  $C_{aa} C_{bb} > C_{ab}^2$ .

charge distribution on the conductors undergoes an infinitesimal change (the total charge on each conductor remaining unaltered), in which the charges may penetrate into the conductors; we ignore the fact that such a charge distribution cannot in reality be stationary. We consider the change in the integral  $\mathcal{U} = (1/8\pi) \int E^2 dV$ , which must now be extended over all space, including the volumes of the conductors themselves (since after the displacement of the charges the field  $\mathbf{E}$  may not be zero inside the conductors). We write

$$\begin{aligned}\delta \mathcal{U} &= -\frac{1}{4\pi} \int \mathbf{grad} \phi \cdot \delta \mathbf{E} dV \\ &= -\frac{1}{4\pi} \int \text{div} (\phi \delta \mathbf{E}) dV + \frac{1}{4\pi} \int \phi \text{div} \delta \mathbf{E} dV.\end{aligned}$$

The first integral vanishes, being equivalent to one over an infinitely remote surface. In the second integral, we have by (1.8)  $\text{div} \delta \mathbf{E} = 4\pi \delta \bar{\rho}$ , so that  $\delta \mathcal{U} = \int \phi \delta \bar{\rho} dV$ . This integral vanishes if  $\phi$  is the potential of the true electrostatic field, since then  $\phi$  is constant inside each conductor, and the integral  $\int \delta \bar{\rho} dV$  over the volume of each conductor is zero, since its total charge remains unaltered.

Thus the energy of the actual electrostatic field is a minimum† relative to the energies of fields which could be produced by any other distribution of the charges on or in the conductors (*Thomson's theorem*).

From this theorem it follows, in particular, that the introduction of an uncharged conductor into the field of given charges (charged conductors) reduces the total energy of the field. To prove this, it is sufficient to compare the energy of the actual field resulting from the introduction of the uncharged conductor with the energy of the fictitious field in which there are no induced charges on that conductor. The former energy, since it has the least possible value, is less than the latter energy, which is also the energy of the original field (since, in the absence of induced charges, the field would penetrate into the conductor, and remain unaltered). This result can also be formulated thus: an uncharged conductor remote from a system of given charges is attracted towards the system.

Finally, it can be shown that a conductor (charged or not) brought into an electrostatic field cannot be in stable equilibrium under electric forces alone. This assertion generalizes the theorem for a point charge proved at the end of §1, and can be derived by combining the latter theorem with Thomson's theorem. We shall not pause to give the derivation in detail.

Formulae (2.9) are useful for calculating the energy of a system of conductors at finite distances apart. The energy of an uncharged conductor in a uniform external field  $\mathfrak{E}$ , which may be imagined as due to charges at infinity, requires special consideration. According to (2.2), this energy is  $\mathcal{U} = \frac{1}{2}e\phi$ , where  $e$  is the remote charge which causes the field, and  $\phi$  is the potential at this charge due to the conductor.  $\mathcal{U}$  does not include the energy of the charge  $e$  in its own field, since we are interested only in the energy of the conductor. The charge on the conductor is zero, but the external field causes it to acquire an electric dipole moment, which we denote by  $\mathcal{P}$ . The potential of the electric dipole field at a large distance  $\mathbf{r}$  from it is  $\phi = \mathcal{P} \cdot \mathbf{r}/r^3$ . Hence  $\mathcal{U} = e \mathcal{P} \cdot \mathbf{r}/2r^3$ . But  $-e\mathbf{r}/r^3$  is just the field  $\mathfrak{E}$  due to the charge  $e$ . Thus

$$\mathcal{U} = -\frac{1}{2} \mathcal{P} \cdot \mathfrak{E}. \quad (2.12)$$

† We shall not give here the simple arguments which demonstrate that the extremum is a minimum.



Since all the field equations are linear, it is evident that the components of the dipole moment  $\mathcal{P}$  are linear functions of the components of the field  $\mathcal{E}$ . The coefficients of proportionality between  $\mathcal{P}$  and  $\mathcal{E}$  have the dimensions of length cubed, and are therefore proportional to the volume of the conductor:

$$\mathcal{P}_i = V\alpha_{ik}\mathcal{E}_k, \quad (2.13)$$

where the coefficients  $\alpha_{ik}$  depend only on the shape of the body. The quantities  $V\alpha_{ik}$  form a tensor, which may be called the *polarizability tensor* of the body. This tensor is symmetrical:  $\alpha_{ik} = \alpha_{ki}$ , a statement which will be proved in §11. Accordingly, the energy (2.12) is

$$\mathcal{U} = -\frac{1}{2}V\alpha_{ik}\mathcal{E}_i\mathcal{E}_k. \quad (2.14)$$

## PROBLEMS

**PROBLEM 1.** Express the mutual capacitance  $C$  of two conductors (with charges  $\pm e$ ) in terms of the coefficients  $C_{ab}$ .

**SOLUTION.** The *mutual capacitance* of two conductors is defined as the coefficient  $C$  in the relation  $e = C(\phi_2 - \phi_1)$ , and the energy of the system is given in terms of  $C$  by  $\mathcal{U} = \frac{1}{2}e^2/C$ . Comparing with (2.9), we obtain

$$\begin{aligned} 1/C &= C^{-1}_{11} - 2C^{-1}_{12} + C^{-1}_{22} \\ &= (C_{11} + 2C_{12} + C_{22})/(C_{11}C_{22} - C_{12}^2). \end{aligned}$$

**PROBLEM 2.** A point charge  $e$  is situated at  $O$ , near a system of earthed conductors, and induces on them charges  $e_a$ . If the charge  $e$  were absent, and the  $a$ th conductor were at potential  $\phi'_a$ , the remainder being earthed, the field potential at  $O$  would be  $\phi'_0$ . Express the charges  $e_a$  in terms of  $\phi'_a$  and  $\phi'_0$ .

**SOLUTION.** If charges  $e_a$  on the conductors give them potentials  $\phi_a$ , and similarly for  $e'_a$  and  $\phi'_a$ , it follows from (2.3) that

$$\sum_a \phi_a e'_a = \sum_{a,b} \phi_a C_{ab} \phi'_b = \sum_a \phi'_a e_a.$$

We apply this relation to two states of the system formed by all the conductors and the charge  $e$  (regarding the latter as a very small conductor). In one state the charge  $e$  is present, the charges on the conductors are  $e_a$ , and their potentials are zero. In the other state the charge  $e$  is zero, and one of the conductors has a potential  $\phi'_a \neq 0$ . Then we have  $e\phi'_0 + e_a\phi'_a = 0$ , whence  $e_a = -e\phi'_0/\phi'_a$ .

For example, if a charge  $e$  is at a distance  $r$  from the centre of an earthed conducting sphere with radius  $a$  ( $< r$ ), then  $\phi'_0 = \phi'_a a/r$ , and the charge induced on the sphere is  $e_a = -ea/r$ .

As a second example, let us consider a charge  $e$  placed between two concentric conducting spheres with radii  $a$  and  $b$ , at a distance  $r$  from the centre such that  $a < r < b$ . If the outer sphere is earthed and the inner one is charged to potential  $\phi'_a$ , the potential at distance  $r$  is

$$\phi'_0 = \phi'_a \frac{1/r - 1/b}{1/a - 1/b}.$$

Hence the charge induced on the inner sphere by the charge  $e$  is  $e_a = -ea(b-r)/r(b-a)$ . Similarly the charge induced on the outer sphere is  $e_b = -eb(r-a)/r(b-a)$ .

**PROBLEM 3.** Two conductors, with capacitances  $C_1$  and  $C_2$ , are placed at a distance  $r$  apart which is large compared with their dimensions. Determine the coefficients  $C_{ab}$ .

**SOLUTION.** If conductor 1 has a charge  $e_1$ , and conductor 2 is uncharged, then in the first approximation  $\phi_1 = e_1/C_1$ ,  $\phi_2 = e_1/r$ ; here we neglect the variation of the field over conductor 2 and its polarization. Thus  $C^{-1}_{11} = 1/C_1$ ,  $C^{-1}_{12} = 1/r$ , and similarly  $C^{-1}_{22} = 1/C_2$ . Hence we find†

$$C_{11} = C_1 \left( 1 + \frac{C_1 C_2}{r^2} \right), \quad C_{12} = -\frac{C_1 C_2}{r}, \quad C_{22} = C_2 \left( 1 + \frac{C_1 C_2}{r^2} \right).$$

† The subsequent terms in the expansion are in general of order (in  $1/r$ ) one higher than those given. If, however,  $r$  is taken as the distance between the "centres of charge" of the two bodies (for spheres, between the geometrical centres), then the order of the subsequent terms is two higher.

**PROBLEM 4.** Determine the capacitance of a ring (radius  $b$ ) of thin conducting wire of circular cross-section (radius  $a \ll b$ ).

**SOLUTION.** Since the wire is thin, the field at the surface of the ring is almost the same as that of charges distributed along the axis of the wire (for a right cylinder, it would be exactly the same). Hence the potential of the ring is

$$\phi_a = \frac{e}{2\pi b} \oint \frac{dl}{r},$$

where  $r$  is the distance from a point on the surface of the ring to an element  $dl$  of the axis of the wire, the integration being over all such elements. We divide the integral into two parts corresponding to  $r < \Delta$  and  $r > \Delta$ ,  $\Delta$  being a distance such that  $a \ll \Delta \ll b$ . Then for  $r < \Delta$  the segment of the ring concerned may be regarded as straight, and therefore

$$\oint_{\Delta > r} \frac{dl}{r} = \int_{-\Delta}^{\Delta} \frac{dl}{\sqrt{l^2 + a^2}} \cong 2 \log(2\Delta/a).$$

In the range  $r > \Delta$  the thickness of the wire may be neglected, i.e.  $r$  may be taken as the distance between two points on its axis. Then

$$\oint_{r > \Delta} \frac{dl}{r} = 2 \int_{\phi_0}^{\pi} \frac{b d\phi}{2b \sin \frac{1}{2}\phi} = -2 \log \tan \frac{1}{4}\phi_0,$$

where  $\phi$  is the angle subtended at the centre of the ring by the chord  $r$ , and the lower limit of integration is such that  $2b \sin \frac{1}{2}\phi_0 = \Delta$ , whence  $\phi_0 \cong \Delta/b$ . When the two parts of the integral are added,  $\Delta$  cancels, and the capacitance of the ring is

$$C = \frac{e}{\phi_a} = \frac{\pi b}{\log(8b/a)}.$$

### §3. Methods of solving problems in electrostatics

The general methods of solving Laplace's equation for given boundary conditions on certain surfaces are studied in mathematical physics, and we shall not give a detailed description of them here. We shall merely mention some of the more elementary procedures and solve various problems of intrinsic interest.†

(1) *The method of images.* The simplest example of the use of this method is to determine the field due to a point charge  $e$  outside a conducting medium which occupies a half-space. The principle of the method is to find fictitious point charges which, together with the given charge or charges, produce a field such that the surface of the conductor is an equipotential surface. In the case just mentioned, this is achieved by placing a fictitious charge  $e' = -e$  at a point which is the image of  $e$  in the plane which bounds the conducting medium. The potential of the field due to the charge  $e$  and its image  $e'$  is

$$\phi = e \left( \frac{1}{r} - \frac{1}{r'} \right), \quad (3.1)$$

† The solutions of many more complex problems are given by W. R. Smythe, *Static and Dynamic Electricity*, 3rd ed., McGraw-Hill, New York, 1968; G. A. Grinberg, *Selected Problems in the Mathematical Theory of Electric and Magnetic Phenomena* (*Izbrannye voprosy matematicheskoi teorii élektricheskikh i magnitnykh yavlenii*), Moscow, 1948.

where  $r$  and  $r'$  are the distances of a point from the charges  $e$  and  $e'$ . On the bounding plane,  $r = r'$  and the potential has the constant value zero, so that the necessary boundary condition is satisfied and (3.1) gives the solution of the problem. It may be noted that the charge  $e$  is attracted to the conductor by a force  $e^2/(2a)^2$  (the *image force*;  $a$  is the distance of the charge from the conductor), and the energy of their interaction is  $-e^2/4a$ .

The distribution of surface charge induced on the bounding plane by the point charge  $e$  is given by

$$\sigma = -\frac{1}{4\pi} \left[ \frac{\partial \phi}{\partial n} \right]_{r=r'} = -\frac{e}{2\pi} \frac{a}{r^3}. \quad (3.2)$$

It is easy to see that the total charge on the plane is  $\int \sigma df = -e$ , as it should be.

The total charge induced on an originally uncharged insulated conductor by other charges is, of course, zero. Hence, if in the present case the conducting medium (in reality a large conductor) is insulated, we must suppose that, besides the charge  $-e$ , a charge  $+e$  is also induced, which, however, has a vanishingly small density, being distributed over the large surface of the conductor.

Next, let us consider a more difficult problem, that of the field due to a point charge  $e$  near a spherical conductor. To solve this problem, we use the following result, which can easily be proved by direct calculation. The potential of the field due to two point charges  $e$  and  $-e'$ , namely  $\phi = e/r - e'/r'$ , vanishes on the surface of a sphere whose centre is on the line joining the charges (but not between them). If the radius of the sphere is  $R$  and its centre is distant  $l$  and  $l'$  from the two charges, then  $l/l' = (e/e')^2$ ,  $R^2 = ll'$ .

Let us first suppose that the spherical conductor is maintained at a constant potential  $\phi = 0$ , i.e. it is earthed. Then the field outside the sphere due to the point charge  $e$  at  $A$  (Fig. 1), at a distance  $l$  from the centre of the sphere, is the same as the field due to two charges, namely the given charge  $e$  and a fictitious charge  $-e'$  at  $A'$  inside the sphere, at a distance  $l'$  from its centre, where

$$l' = R^2/l, \quad e' = eR/l. \quad (3.3)$$

The potential of this field is

$$\phi = \frac{e}{r} - \frac{eR}{lr'}, \quad (3.4)$$

$r$  and  $r'$  being as shown in Fig. 1. A non-zero total charge  $-e'$  is induced on the surface of the sphere. The energy of the interaction between the charge and the sphere is

$$\mathcal{U} = -\frac{1}{2}ee'/(l-l') = -\frac{1}{2}e^2R/(l^2 - R^2), \quad (3.5)$$

and the charge is attracted to the sphere by a force  $F = -\partial \mathcal{U} / \partial l = -e^2lR/(l^2 - R^2)^2$ .

If the total charge on the spherical conductor is kept equal to zero (an insulated uncharged sphere), a further fictitious charge must be introduced, such that the total charge induced on the surface of the sphere is zero, and the potential on that surface is still constant. This is done by placing a charge  $+e'$  at the centre of the sphere. The potential of the required field is then given by the formula

$$\phi = \frac{e}{r} - \frac{e'}{r'} + \frac{e'}{r_0}. \quad (3.6)$$

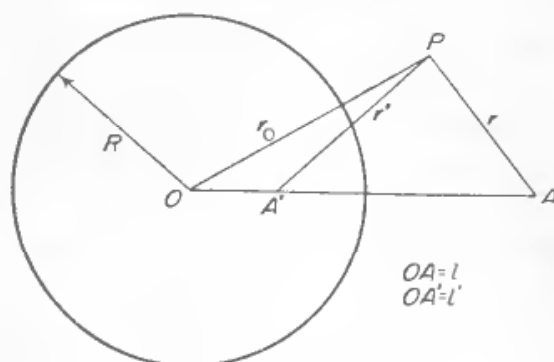


FIG. 1

The energy of interaction in this case is

$$U = \frac{1}{2}ee' \left( \frac{1}{l} - \frac{1}{l-l'} \right) = -\frac{e^2 R^3}{2l^2(l^2 - R^2)}. \quad (3.7)$$

Finally, if the charge  $e$  is at  $A'$  (Fig. 1) in a spherical cavity in a conducting medium, the field inside the cavity must be the same as the field due to the charge  $e$  at  $A'$  and its image at  $A$  outside the sphere, regardless of whether the conductor is earthed or insulated:

$$\phi = \frac{e}{r'} - \frac{eR}{l'r}. \quad (3.8)$$

(2) *The method of inversion.* There is a simple method whereby in some cases a known solution of one electrostatic problem gives the solution of another problem. This method is based on the invariance of Laplace's equation with respect to a certain transformation of the variables.

In spherical polar coordinates Laplace's equation has the form

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \Delta_{\Omega} \phi = 0,$$

where  $\Delta_{\Omega}$  denotes the angular part of the Laplacian operator. It is easy to see that this equation is unaltered in form if the variable  $r$  is replaced by a new variable  $r'$  such that

$$r = R^2/r' \quad (3.9)$$

(the *inversion* transformation) and at the same time the unknown function  $\phi$  is replaced by  $\phi'$  such that

$$\phi = r' \phi' / R. \quad (3.10)$$

Here  $R$  is some constant having the dimensions of length (the *radius of inversion*). Thus, if the function  $\phi(\mathbf{r})$  satisfies Laplace's equation, then so does the function

$$\phi'(\mathbf{r}') = R \phi(R^2 \mathbf{r}' / r'^2) / r'. \quad (3.11)$$

Let us assume that we know the electrostatic field due to some system of conductors, all at the same potential  $\phi_0$ , and point charges. The potential  $\phi(\mathbf{r})$  is usually defined so as to vanish at infinity. Here, however, we shall define  $\phi(\mathbf{r})$  so that it tends to  $-\phi_0$  at infinity. Then  $\phi = 0$  on the conductors.

We may now ascertain what problem of electrostatics will be solved by the transformed function (3.11). First of all, the shapes and relative positions of all the conductors of finite size will be changed. The boundary condition of constant potential on their surfaces will be automatically satisfied, since  $\phi' = 0$  if  $\phi = 0$ . Furthermore, the positions and magnitudes of all the point charges will be changed. A charge  $e$  at a point  $\mathbf{r}_0$  moves to  $\mathbf{r}'_0 = R^2\mathbf{r}_0/r_0^2$  and takes a value  $e'$  which can be determined as follows. As  $\mathbf{r} \rightarrow \mathbf{r}_0$  the potential  $\phi(\mathbf{r})$  tends to infinity as  $e/|\delta\mathbf{r}|$ , where  $\delta\mathbf{r} = \mathbf{r} - \mathbf{r}_0$ . Differentiating the relation  $\mathbf{r} = R^2\mathbf{r}'/r'^2$ , we find that the magnitudes of the small differences  $\delta\mathbf{r}$  and  $\delta\mathbf{r}' = \mathbf{r}' - \mathbf{r}'_0$  are related by  $(\delta\mathbf{r})^2 = R^4(\delta\mathbf{r}')^2/r_0'^4$ . Hence, as  $\mathbf{r}' \rightarrow \mathbf{r}'_0$ , the function  $\phi'$  tends to infinity as  $eR/r'_0|\delta\mathbf{r}| = e'r'_0/R|\delta\mathbf{r}'|$ , corresponding to a charge

$$e' = er'_0/R = eR/r_0. \quad (3.12)$$

Finally, let us examine the behaviour of the function  $\phi'(\mathbf{r}')$  near the origin. For  $\mathbf{r}' = 0$  we have  $\mathbf{r} \rightarrow \infty$  and  $\phi(\mathbf{r}) \rightarrow -\phi_0$ . Hence, as  $\mathbf{r}' \rightarrow 0$ , the function  $\phi'$  tends to infinity as  $-R\phi_0/r'$ . This means that there is a charge  $e_0 = -R\phi_0$  at the point  $\mathbf{r}' = 0$ .

We shall give, for reference, the way in which certain geometrical figures are transformed by inversion. A spherical surface with radius  $a$  and centre  $\mathbf{r}_0$  is given by the equation  $(\mathbf{r} - \mathbf{r}_0)^2 = a^2$ . On inversion, this becomes  $([R^2\mathbf{r}'/r'^2] - \mathbf{r}_0)^2 = a^2$ , which, on multiplying by  $r'^2$  and rearranging, can be written  $(\mathbf{r}' - \mathbf{r}'_0)^2 = a'^2$ , where

$$\mathbf{r}'_0 = -R^2\mathbf{r}_0/(a^2 - r_0^2), \quad a' = aR^2/|a^2 - r_0^2|. \quad (3.13)$$

Thus we have another sphere, with radius  $a'$  and centre  $\mathbf{r}'_0$ . If the original sphere passes through the origin ( $a = r_0$ ), then  $a' = \infty$ . In this case the sphere is transformed into a plane perpendicular to the vector  $\mathbf{r}_0$  and distant  $r'_0 - a' = R^2/(a + r_0) = R^2/2a$  from the origin.

(3) *The method of conformal mapping.* A field which depends on only two Cartesian co-ordinates ( $x$  and  $y$ , say) is said to be *two-dimensional*. The theory of functions of a complex variable is a powerful means of solving two-dimensional problems of electrostatics. The theoretical basis of the method is as follows.

An electrostatic field in a vacuum satisfies two equations:  $\text{curl } \mathbf{E} = 0$ ,  $\text{div } \mathbf{E} = 0$ . The first of these makes it possible to introduce the field potential, defined by  $\mathbf{E} = -\text{grad } \phi$ . The second equation shows that we can also define a *vector potential*  $\mathbf{A}$  of the field, such that  $\mathbf{E} = \text{curl } \mathbf{A}$ . In the two-dimensional case, the vector  $\mathbf{E}$  lies in the  $xy$ -plane, and depends only on  $x$  and  $y$ . Accordingly, the vector  $\mathbf{A}$  can be chosen so that it is perpendicular to the  $xy$ -plane. Then the field components are given in terms of the derivatives of  $\phi$  and  $\mathbf{A}$  by

$$E_x = -\partial\phi/\partial x = \partial A/\partial y, \quad E_y = -\partial\phi/\partial y = -\partial A/\partial x. \quad (3.14)$$

These relations between the derivatives of  $\phi$  and  $A$  are, mathematically, just the well-known Cauchy-Riemann conditions, which express the fact that the complex quantity

$$w = \phi - iA \quad (3.15)$$

is an analytic function of the complex argument  $z = x + iy$ . This means that the function  $w(z)$  has a definite derivative at every point, independent of the direction in which the derivative is taken. For example, differentiating along the  $x$ -axis, we find  $dw/dz = \partial\phi/\partial x - i\partial A/\partial x$ , or

$$dw/dz = -E_x + iE_y. \quad (3.16)$$

The function  $w$  is called the *complex potential*.

The lines of force are defined by the equation  $dx/E_x = dy/E_y$ . Expressing  $E_x$  and  $E_y$  as derivatives of  $A$ , we can write this as  $(\partial A/\partial x)dx + (\partial A/\partial y)dy = dA = 0$ , whence  $A(x, y) = \text{constant}$ . Thus the lines on which the imaginary part of the function  $w(z)$  is constant are the lines of force. The lines on which its real part is constant are the equipotential lines. The orthogonality of these families of lines is ensured by the relations (3.14), according to which

$$\frac{\partial \phi}{\partial x} \frac{\partial A}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial A}{\partial y} = 0.$$

Both the real and the imaginary part of an analytic function  $w(z)$  satisfy Laplace's equation. We could therefore equally well take  $\text{im } w$  as the field potential. The lines of force would then be given by  $\text{re } w = \text{constant}$ . Instead of (3.15) we should have  $w = A + i\phi$ .

The flux of the electric field through any section of an equipotential line is given by the integral  $\oint E_n dl = -\oint (\partial \phi/\partial n) dl$ , where  $dl$  is an element of length of the equipotential line and  $n$  the direction of the normal to it. According to (3.14) we have  $\partial \phi/\partial n = -\partial A/\partial l$ , the choice of sign denoting that  $l$  is measured to the left when one looks along  $n$ . Thus  $\oint E_n dl = \oint (\partial A/\partial l) dl = A_2 - A_1$ , where  $A_2$  and  $A_1$  are the values of  $A$  at the ends of the section. In particular, since the flux of the electric field through a closed contour is  $4\pi e$ , where  $e$  is the total charge enclosed by the contour (per unit length of conductors perpendicular to the plane), it follows that

$$e = (1/4\pi)\Delta A, \quad (3.17)$$

where  $\Delta A$  is the change in  $A$  on passing counterclockwise round the closed equipotential line.

The simplest example of the complex potential is that of the field of a charged straight wire passing through the origin and perpendicular to the plane. The field is given by  $E_r = 2e/r$ ,  $E_\theta = 0$ , where  $r, \theta$  are polar coordinates in the  $xy$ -plane, and  $e$  is the charge per unit length of the wire. The corresponding complex potential is

$$w = -2e \log z = -2e \log r - 2ie\theta. \quad (3.18)$$

If the charged wire passes through the point  $(x_0, y_0)$  instead of the origin, the complex potential is

$$w = -2e \log(z - z_0), \quad (3.19)$$

where  $z_0 = x_0 + iy_0$ .

Mathematically, the functional relation  $w = w(z)$  constitutes a *conformal mapping* of the plane of the complex variable  $z$  on the plane of the complex variable  $w$ . Let  $C$  be the cross-sectional contour of a conductor in the  $xy$ -plane, and  $\phi_0$  its potential. It is clear from the above discussion that the problem of determining the field due to this conductor amounts to finding a function  $w(z)$  which maps the contour  $C$  in the  $z$ -plane on the line  $w = \phi_0$ , parallel to the axis of ordinates, in the  $w$ -plane. Then  $\text{re } w$  gives the potential of the field. (If the function  $w(z)$  maps the contour  $C$  on a line parallel to the axis of abscissae, then the potential is  $\text{im } w$ .)

(4) *The wedge problem.* We shall give here, for reference, formulae for the field due to a point charge  $e$  placed between two intersecting conducting half-planes. Let the  $z$ -axis of a system of cylindrical polar coordinates  $(r, \theta, z)$  be along the apex of the wedge, the angle  $\theta$

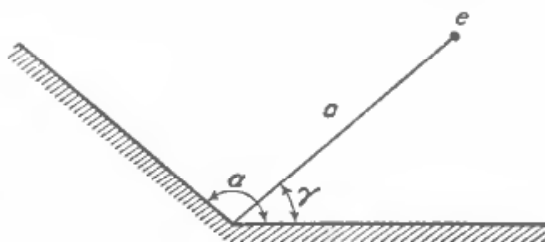


FIG. 2

being measured from one of the planes, and let the position of the charge  $e$  be  $(a, \gamma, 0)$  (Fig. 2). The angle  $\alpha$  between the planes may be either less or greater than  $\pi$ ; in the latter case we have a charge outside a conducting wedge.

The field potential is given by

$$\phi = \frac{e}{\alpha\sqrt{2ar}} \int_{\eta}^{\infty} \left\{ \frac{\sinh(\pi\zeta/\alpha)}{\cosh(\pi\zeta/\alpha) - \cos[\pi(\theta - \gamma)/\alpha]} - \frac{\sinh(\pi\zeta/\alpha)}{\cosh(\pi\zeta/\alpha) - \cos[\pi(\theta + \gamma)/\alpha]} \right\} \times \frac{d\zeta}{\sqrt{(\cosh \zeta - \cosh \eta)}}, \quad \cosh \eta = (a^2 + r^2 + z^2)/2ar, \quad \eta > 0. \quad (3.20)$$

The potential  $\phi = 0$  on the surface of the conductors, i.e. for  $\theta = 0$  or  $\alpha$ . This formula was first given by H. M. Macdonald (1895)†.

In particular, for  $\alpha = 2\pi$  we have a conducting half-plane in the field of a point charge. In this case the integral in (3.20) can be evaluated explicitly, giving

$$\left. \begin{aligned} \phi &= \frac{e}{\pi} \left\{ \frac{1}{R} \cos^{-1} \left( \frac{-\cos \frac{1}{2}(\theta - \gamma)}{\cosh \frac{1}{2}\eta} \right) - \frac{1}{R'} \cos^{-1} \left( \frac{-\cos \frac{1}{2}(\theta + \gamma)}{\cosh \frac{1}{2}\eta} \right) \right\}, \\ R^2 &= a^2 + r^2 + z^2 - 2ar \cos(\gamma - \theta), \\ R'^2 &= a^2 + r^2 + z^2 - 2ar \cos(\gamma + \theta). \end{aligned} \right\} \quad (3.21)$$

In the limit as the point  $(r, \theta, z)$  tends to the position of the charge  $e$ , the potential (3.21) becomes

$$\phi = \phi' + e/R, \quad \text{where } \phi' = -\frac{e}{2\pi a} \left[ 1 + \frac{\pi - \gamma}{\sin \gamma} \right]. \quad (3.22)$$

The second term is just the Coulomb potential, which becomes infinite as  $R \rightarrow 0$ , while  $\phi'$  is the change caused by the conductor in the potential at the position of the charge. The energy of the interaction between the charge and the conducting half-plane is

$$\mathcal{U} = \frac{1}{2}e\phi' = -\frac{e^2}{4\pi a} \left[ 1 + \frac{\pi - \gamma}{\sin \gamma} \right]. \quad (3.23)$$

† Its derivation is given by him in *Electromagnetism*, Bell, London, 1934, p. 79, and by V. V. Batygin and I. N. Toptygin, *Problems in Electrodynamics*, 2nd ed., Academic Press, London, 1978, p. 47.

## PROBLEMS

**PROBLEM 1.** Determine the field near an uncharged conducting sphere with radius  $R$  placed in a uniform external electric field  $\mathcal{E}$ .

**SOLUTION.** We write the potential in the form  $\phi = \phi_0 + \phi_1$ , where  $\phi_0 = -\mathcal{E} \cdot \mathbf{r}$  is the potential of the external field and  $\phi_1$  is the required change in potential due to the sphere. By symmetry, the function  $\phi_1$  can depend only on the constant vector  $\mathcal{E}$ . The only such solution of Laplace's equation which vanishes at infinity is

$$\phi_1 = -\text{constant} \times \mathcal{E} \cdot \text{grad}(1/r) = \text{constant} \times \mathcal{E} \cdot \mathbf{r}/r^3,$$

the origin being taken at the centre of the sphere. On the surface of the sphere  $\phi$  must be constant, and so the constant in  $\phi_1$  is  $R^3$ , whence

$$\phi = -\mathcal{E} r \cos \theta \left( 1 - \frac{R^3}{r^3} \right),$$

where  $\theta$  is the angle between  $\mathcal{E}$  and  $\mathbf{r}$ . The distribution of charge on the surface of the sphere is given by

$$\sigma = -(1/4\pi)[\partial\phi/\partial r]_{r=R} = (3\mathcal{E}/4\pi) \cos \theta.$$

The total charge  $e = 0$ . The dipole moment of the sphere is most easily found by comparing  $\phi_1$  with the potential  $\mathcal{P} \cdot \mathbf{r}/r^3$  of an electric dipole field, whence  $\mathcal{P} = R^3 \mathcal{E}$ .

**PROBLEM 2.** The same as Problem 1, but for an infinite cylinder in a uniform transverse field.

**SOLUTION.** We use polar coordinates in a plane perpendicular to the axis of the cylinder. The solution of the two-dimensional Laplace's equation which depends only on a constant vector is

$$\phi_1 = \text{constant} \times \mathcal{E} \cdot \text{grad}(\log r) = \text{constant} \times \mathcal{E} \cdot \mathbf{r}/r^2.$$

Adding  $\phi_0 = -\mathcal{E} \cdot \mathbf{r}$  and putting the constant equal to  $R^2$ , we have

$$\phi = -\mathcal{E} r \cos \theta \left( 1 - \frac{R^2}{r^2} \right).$$

The surface charge density is  $\sigma = (\mathcal{E}/2\pi) \cos \theta$ . The dipole moment per unit length of the cylinder can be found by comparing  $\phi$  with the potential of a two-dimensional dipole field, namely  $2\mathcal{P} \cdot \text{grad}(\log r) = 2\mathcal{P} \cdot \mathbf{r}/r^2$ , so that  $\mathcal{P} = \frac{1}{2}R^2 \mathcal{E}$ .

**PROBLEM 3.** Determine the field near a wedge-shaped projection on a conductor.

**SOLUTION.** We take polar coordinates  $r, \theta$  in a plane perpendicular to the apex of the wedge, the origin being at the vertex of the angle  $\theta_0$  of the wedge (Fig. 3). The angle  $\theta$  is measured from one face of the wedge, the region outside the conductor being  $0 \leq \theta \leq 2\pi - \theta_0$ . Near the apex of the wedge the potential can be expanded in powers of  $r$ , and we shall be interested in the first term of the expansion (after the constant term), which contains the lowest power of  $r$ . The solutions of the two-dimensional Laplace's equation which are proportional to  $r^n$  are  $r^n \cos n\theta$  and  $r^n \sin n\theta$ . The solution having the smallest  $n$  which satisfies the condition  $\phi = \text{constant}$  for  $\theta = 0$  and  $\theta = 2\pi - \theta_0$  (i.e. on the surface of the conductor) is

$$\phi = \text{constant} \times r^n \sin n\theta, \quad n = \pi/(2\pi - \theta_0).$$

The field varies as  $r^{n-1}$ . For  $\theta_0 < \pi$  ( $n < 1$ ), therefore, the field becomes infinite at the apex of the wedge. In particular, for a very sharp wedge ( $\theta_0 \ll 1$ ,  $n \cong \frac{1}{2}$ )  $E$  increases as  $r^{-1/2}$  as  $r \rightarrow 0$ . Near a wedge-shaped concavity in a conductor ( $\theta_0 > \pi$ ,  $n > 1$ ) the field tends to zero.

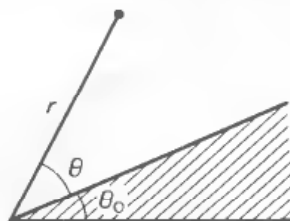


FIG. 3



The value of the constant can be determined only by solving the problem for the whole field. For example, for a very sharp wedge in the field of a point charge  $e$ , the passage to the limit of small  $r$  in (3.21) confirms that

$$\phi = \text{constant} \times \sqrt{r} \sin \frac{1}{2}\theta,$$

the constant being  $[4e\sqrt{a/\pi(a^2+z^2)}] \sin \frac{1}{2}\gamma$ . In this case, "near the wedge" means that  $r \ll a$ , under which condition the  $\partial^2\phi/\partial z^2$  term in Laplace's equation may be neglected.

**PROBLEM 4.** Determine the field near the end of a sharp conical point on the surface of a conductor.

**SOLUTION.** We take spherical polar coordinates, with the origin at the vertex of the cone and the polar axis along the axis of the cone. Let the angle of the cone be  $2\theta_0 \leq 1$ , so that the region outside the conductor corresponds to polar angles in the range  $\theta_0 \leq \theta \leq \pi$ . As in Problem 3, we seek a solution for the variable part of the potential, which is symmetrical about the axis, in the form

$$\phi = r^n f(\theta), \quad (1)$$

with the smallest possible value of  $n$ . Laplace's equation

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) = 0,$$

after substitution of (1), gives

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{df}{d\theta} \right) + n(n+1)f = 0. \quad (2)$$

The condition of constant potential on the surface of the cone means that we must have  $f(\theta_0) = 0$ .

For small  $\theta_0$  we seek a solution by assuming that  $n \ll 1$  and  $f(\theta)$  is of the form  $\text{constant} \times [1 + \psi(\theta)]$ , where  $\psi \ll 1$ . (For  $\theta_0 \rightarrow 0$ , i.e. an infinitely sharp point, we should expect that  $\phi$  tends to a constant almost everywhere near the cone.) The equation for  $\psi$  is

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\psi}{d\theta} \right) = -n. \quad (3)$$

The solution having no singularities outside the cone (in particular, at  $\theta = \pi$ ) is  $\psi(\theta) = 2n \log \sin \frac{1}{2}\theta$ .

For  $\theta \sim \theta_0 \ll 1$ ,  $\psi$  is no longer small. Nevertheless, this expression remains valid, since the second term in equation (2) may be neglected because  $\theta$  is small. To determine the constant  $n$  in the first approximation we must require that the function  $f = 1 + \psi$  vanish for  $\theta = \theta_0$ . Thus  $n = -1/2 \log \theta_0$ . The field increases to infinity as  $r^{-(1-n)}$  in the neighbourhood of the vertex, i.e. essentially as  $1/r$ .

**PROBLEM 5.** The same as Problem 4, but for a sharp conical depression on the surface of a conductor.

**SOLUTION.** The region outside the conductor now corresponds to the range  $0 \leq \theta \leq \theta_0$ . As in Problem 4, we seek  $\phi$  in the form (1), but now  $n \gg 1$ . Since  $\theta \ll 1$  for all points in the field, equation (2) becomes

$$\frac{1}{\theta} \frac{d}{d\theta} \left( \theta \frac{df}{d\theta} \right) + n^2 f = 0.$$

This is Bessel's equation, and the solution having no singularities in the field is  $J_0(n\theta)$ . The value of  $n$  is determined as the smallest root of the equation  $J_0(n\theta_0) = 0$ , whence  $n = 2.4/\theta_0$ .

**PROBLEM 6.** Determine the energy of the attraction between an electric dipole and a plane conducting surface.

**SOLUTION.** We take the  $x$ -axis perpendicular to the surface of the conductor, and passing through the dipole; let the dipole moment vector  $\mathcal{P}$  lie in the  $xy$ -plane. The image of the dipole is at the point  $-x$  and has a moment  $\mathcal{P}'_x = \mathcal{P}_x$ ,  $\mathcal{P}'_y = -\mathcal{P}_y$ . The required energy of attraction is the energy of the interaction between the dipole and its image, and is  $\mathcal{U} = -(2\mathcal{P}_x^2 + \mathcal{P}_y^2)/8x^3$ .

**PROBLEM 7.** Determine the mutual capacitance per unit length of two parallel infinite conducting cylinders with radii  $a$  and  $b$ , their axes being at a distance  $c$  apart.†

† A more exact formula  $n = 1/2 \log(2/\theta_0)$ , containing a coefficient in the (large) logarithm, cannot really be obtained by the simple method given here. A more rigorous calculation, however, leads, as it happens, to this same formula.

‡ The corresponding problem for two spheres cannot be solved in closed form. The difference arises because, in the field of two parallel wires bearing equal and opposite charges, all the equipotential surfaces are circular cylinders, whereas in the field of two equal and opposite point charges the equipotential surfaces are not spheres.

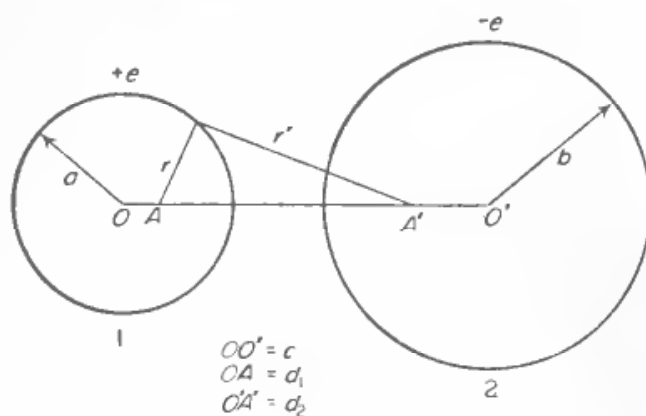


FIG. 4

**SOLUTION.** The field due to the two cylinders is the same as that which would be produced (in the region outside the cylinders) by two charged wires passing through certain points  $A$  and  $A'$  (Fig. 4). The wires have charges  $\pm e$  per unit length, equal to the charges on the cylinders, and the points  $A$  and  $A'$  lie on  $OO'$  in such a way that the surfaces of the cylinders are equipotential surfaces. For this to be so, the distances  $OA$  and  $O'A'$  must be such that  $OA \cdot OA' = a^2$ ,  $O'A' \cdot O'A = b^2$ , i.e.  $d_1(c - d_2) = a^2$ ,  $d_2(c - d_1) = b^2$ . Then, for each cylinder, the ratio  $r/r'$  of the distances from  $A$  and  $A'$  is constant. On cylinder 1,  $r/r' = a/OA' = a/(c - d_2) = d_1/a$ , and on cylinder 2,  $r'/r = d_2/b$ . Accordingly, the potentials of the cylinders are  $\phi_1 = -2e \log(r/r') = -2e \log(d_1/a)$ ,  $\phi_2 = 2e \log(d_2/b)$ ,  $\phi_2 - \phi_1 = 2e \log(d_1 d_2 / ab)$ . Hence we find the required mutual capacitance  $C = e/(\phi_2 - \phi_1)$ :

$$1/C = 2 \log(d_1 d_2 / ab) = 2 \cosh^{-1} [(c^2 - a^2 - b^2)/2ab].$$

In particular, for a cylinder with radius  $a$  at a distance  $h (> a)$  from a conducting plane, we put  $c = b + h$  and take the limit as  $b \rightarrow \infty$ , obtaining  $1/C = 2 \cosh^{-1}(h/a)$ .

If two hollow cylinders are placed one inside the other ( $c < b - a$ ), there is no field outside, while the field between the cylinders is the same as that due to two wires with charges  $\pm e$  passing through  $A$  and  $A'$  (Fig. 5). The same method gives

$$1/C = 2 \cosh^{-1} [(a^2 + b^2 - c^2)/2ab].$$

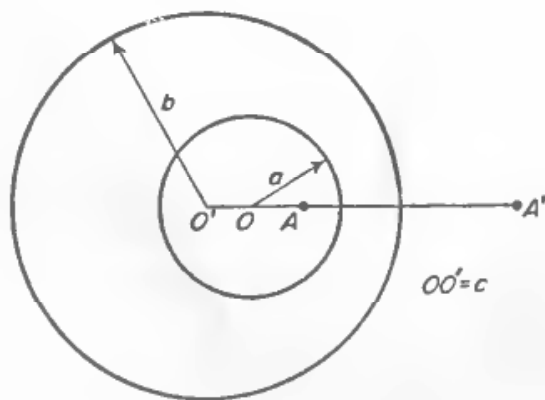


FIG. 5

**PROBLEM 8.** The boundary of a conductor is an infinite plane with a hemispherical projection. Determine the charge distribution on the surface.

**SOLUTION.** In the field determined in Problem 1, whose potential is

$$\phi = \text{constant} \times z \left( 1 - \frac{R^3}{r^3} \right),$$

the plane  $z = 0$  with a projection  $r = R$  is an equipotential surface, on which  $\phi = 0$ . Hence it can be the surface of a conductor, and the above formula gives the field outside the conductor. The charge distribution on the plane part of the surface is given by

$$\sigma = -\frac{1}{4\pi} \left[ \frac{\partial \phi}{\partial z} \right]_{z=0} = \sigma_0 \left( 1 - \frac{R^3}{r^3} \right);$$

we have taken the constant in  $\phi$  as  $-4\pi\sigma_0$ , so that  $\sigma_0$  is the charge density far from the projection. On the surface of the projection we have

$$\sigma = -\frac{1}{4\pi} \left[ \frac{\partial \phi}{\partial r} \right]_{r=R} = 3\sigma_0 \frac{z}{R}.$$

**PROBLEM 9.** Determine the dipole moment of a thin conducting cylindrical rod, with length  $2l$  and radius  $a \ll l$ , in an electric field  $\mathcal{E}$  parallel to its axis.

**SOLUTION.** Let  $\tau(z)$  be the charge per unit length induced on the surface of the rod, and  $z$  the coordinate along the axis of the rod, measured from its midpoint. The condition of constant potential on the surface of the conductor is

$$-\mathcal{E}z + \frac{1}{2\pi} \int_0^{2\pi} \int_{-l}^l \frac{\tau(z') dz' d\phi}{R} = 0,$$

$$R^2 = (z' - z)^2 + 4a^2 \sin^2 \frac{1}{2}\phi,$$

where  $\phi$  is the angle between planes passing through the axis of the cylinder and through two points on its surface at a distance  $R$  apart. We divide the integral into two parts, putting  $\tau(z') \equiv \tau(z) + [\tau(z') - \tau(z)]$ . Since  $l \gg a$ , we have for points not too near the ends of the rod

$$\frac{\tau(z)}{2\pi} \iint \frac{dz' d\phi}{R} \cong \frac{\tau(z)}{2\pi} \int_0^{2\pi} \log \frac{l^2 - z^2}{a^2 \sin^2 \frac{1}{2}\phi} d\phi = \tau(z) \log \frac{4(l^2 - z^2)}{a^2},$$

using the result that  $\int_0^\pi \log \sin \phi d\phi = -\pi \log 2$ . In the integral which contains the difference  $\tau(z') - \tau(z)$ , we can neglect the  $a^2$  term in  $R$ , since it no longer causes the integral to diverge. Thus

$$\mathcal{E}z = \tau(z) \log 4(l^2 - z^2)/a^2 + \int_{-l}^l \frac{\tau(z') - \tau(z)}{|z' - z|} dz'.$$

The quantity  $\tau$  is almost proportional to  $z$ , and in this approximation the integral gives  $-2\tau(z)$ , the result being

$$\tau(z) = \frac{\mathcal{E}z}{\log [4(l^2 - z^2)/a^2] - 2}.$$

This expression is invalid near the ends of the rod, but in calculating the dipole moment that region is unimportant. In the above approximation we have

$$\begin{aligned} \mathcal{P} &= \int_{-l}^l \tau(z) z dz = \frac{\mathcal{E}}{L} \int_0^l \left\{ z^2 - \frac{z^2}{2L} \log \left( 1 - \frac{z^2}{l^2} \right) \right\} dz \\ &= \frac{\mathcal{E}l^3}{3L} \left\{ 1 + \frac{1}{L} \left( \frac{4}{3} - \log 2 \right) \right\}, \end{aligned}$$

where  $L = \log(2l/a) - 1$  is large, or (with the same accuracy)

$$\mathcal{P} = \frac{\mathcal{E}l^3}{3 \log(4l/a) - 7}.$$

**PROBLEM 10.** Determine the capacitance of a hollow conducting cap of a sphere.

**SOLUTION.** We take the origin  $O$  at a point on the rim of the cap (Fig. 6), and carry out the inversion transformation  $r = l^2/r'$ , where  $l$  is the diameter of the cap. The cap then becomes the half-plane shown by the dashed line in Fig. 6, which is perpendicular to the radius  $AO$  of the cap and passes through the point  $B$  on its rim.

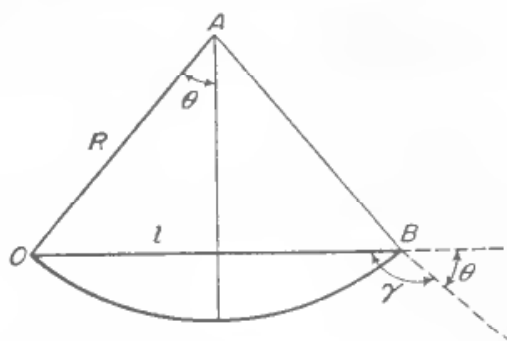


FIG. 6

The angle  $\gamma = \pi - \theta$ , where  $2\theta$  is the angle subtended by the diameter of the cap at the centre of the sphere.

If the charge on the cap is  $e$  and its potential is taken as zero, then as  $r \rightarrow \infty$  the potential  $\phi \rightarrow -\phi_0 + e/r$ . Accordingly, in the transformed problem, as  $r' \rightarrow 0$  the potential is  $\phi' \rightarrow l\phi/r' \cong -l\phi_0/r' + e/l$ , where the first term corresponds to a charge  $e' = -l\phi_0$  at the origin.

According to formula (3.22) we have

$$\phi' = \frac{e'}{r'} - \frac{e'}{2\pi l} \left( 1 + \frac{\theta}{\sin \theta} \right)$$

(the potential near a charge  $e'$  at a distance  $l$  from the edge of a conducting half-plane at zero potential). Comparing the two expressions, we have for the required capacitance  $C = e/\phi_0$

$$C = \frac{l}{2\pi} \left( 1 + \frac{\theta}{\sin \theta} \right) = \frac{R}{\pi} (\sin \theta + \theta),$$

where  $R$  is the radius of the cap.

**PROBLEM 11.** Determine the correction due to edge effects on the value  $C = S/4\pi d$  for the capacitance of a plane capacitor ( $S$  being the area of the plates, and  $d \ll \sqrt{S}$  the distance between them).

**SOLUTION.** Since the plates have free edges, the distribution of charge over them is not uniform. To determine the required correction in a first approximation, we consider points which are at distances  $x$  from the edge such that  $d \ll x \ll \sqrt{S}$ . For example, taking the upper layer (at potential  $\phi = \frac{1}{2}\phi_0$ , Fig. 7a) and neglecting its distance  $\frac{1}{2}d$  from the midplane (the equipotential surface  $\phi = 0$ ), we have the problem of the field near the boundary between two parts of a plane having different potentials (Fig. 7b). The solution is elementary†, and the excess charge density (relative to the value of  $\sigma$  far from the edge) is  $\Delta\sigma = E_n/4\pi = \phi_0/8\pi^2 x$ , so that the total excess charge is  $L \int \Delta\sigma dx = (\phi_0 L/8\pi^2) \log(\sqrt{S}/d)$ , where  $L$  is the perimeter of the plate. In calculating the logarithmically divergent integral, we have taken the limits as those of the region  $d \ll x \ll \sqrt{S}$ . Hence we find the capacitance

$$C = \frac{S}{4\pi d} + \frac{L}{8\pi^2} \log \frac{\sqrt{S}}{d}.$$

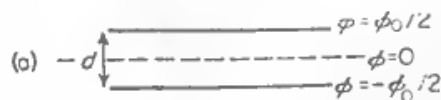


FIG. 7

† See §23. In formula (23.2) for the potential we must here put  $\phi_{ab} = \frac{1}{2}\phi_0$ ,  $\alpha = \pi$ .

A more exact calculation (determining the coefficient in the argument of the logarithm) demands considerably more elaborate methods, and the result depends the shape of the plates. If these are circular, with radius  $R$ , we obtain Kirchhoff's formula

$$C = \frac{R^2}{4d} + \frac{R}{4\pi} \left( \log \frac{16\pi R}{d} - 1 \right).$$

#### §4. A conducting ellipsoid

The problem of the field of a charged conducting ellipsoid and that of an ellipsoid in a uniform external field are solved by the use of *ellipsoidal coordinates*. These are related to Cartesian coordinates by the equation

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1 \quad (a > b > c). \quad (4.1)$$

This equation, a cubic in  $u$ , has three different real roots  $\xi, \eta, \zeta$ , which lie in the following ranges:

$$\xi \geq -c^2, \quad -c^2 \geq \eta \geq -b^2, \quad -b^2 \geq \zeta \geq -a^2. \quad (4.2)$$

These three roots are the ellipsoidal coordinates of the point  $x, y, z$ . Their geometrical significance is seen from the fact that the surfaces of constant  $\xi, \eta$  and  $\zeta$  are respectively ellipsoids and hyperboloids of one and two sheets, all confocal with the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1. \quad (4.3)$$

One surface of each of the three families passes through each point in space, and the three surfaces are orthogonal. The formulae for transformation from ellipsoidal to Cartesian coordinates are given by solving three simultaneous equations of the type (4.1), and are

$$\left. \begin{aligned} x &= \pm \sqrt{\left[ \frac{(\xi + a^2)(\eta + a^2)(\zeta + a^2)}{(b^2 - a^2)(c^2 - a^2)} \right]}, \\ y &= \pm \sqrt{\left[ \frac{(\xi + b^2)(\eta + b^2)(\zeta + b^2)}{(c^2 - b^2)(a^2 - b^2)} \right]}, \\ z &= \pm \sqrt{\left[ \frac{(\xi + c^2)(\eta + c^2)(\zeta + c^2)}{(a^2 - c^2)(b^2 - c^2)} \right]}. \end{aligned} \right\} \quad (4.4)$$

The element of length in ellipsoidal coordinates is

$$\left. \begin{aligned} dl^2 &= h_1^2 d\xi^2 + h_2^2 d\eta^2 + h_3^2 d\zeta^2, \\ h_1 &= \sqrt{[(\xi - \eta)(\xi - \zeta)]/2R_\xi}, \quad h_2 = \sqrt{[(\eta - \zeta)(\eta - \xi)]/2R_\eta}, \\ h_3 &= \sqrt{[(\zeta - \xi)(\zeta - \eta)]/2R_\zeta}, \quad R_u^2 = (u + a^2)(u + b^2)(u + c^2), \\ &\quad u = \xi, \eta, \zeta. \end{aligned} \right\} \quad (4.5)$$

Accordingly, Laplace's equation in these coordinates is

$$\Delta \phi = \frac{4}{(\xi - \eta)(\zeta - \xi)(\eta - \zeta)} \times \left[ (\eta - \zeta)R_\xi \frac{\partial}{\partial \xi} \left( R_\xi \frac{\partial \phi}{\partial \xi} \right) + (\zeta - \xi)R_\eta \frac{\partial}{\partial \eta} \left( R_\eta \frac{\partial \phi}{\partial \eta} \right) + (\xi - \eta)R_\zeta \frac{\partial}{\partial \zeta} \left( R_\zeta \frac{\partial \phi}{\partial \zeta} \right) \right] = 0. \quad (4.6)$$

If two of the semiaxes  $a, b, c$  become equal, the system of ellipsoidal coordinates degenerates. Let  $a = b > c$ . Then the cubic equation (4.1) becomes a quadratic,

$$\frac{\rho^2}{a^2 + u} + \frac{z^2}{c^2 + u} = 1, \quad \rho^2 = x^2 + y^2, \quad (4.7)$$

with two roots whose values lie in the ranges  $\xi \geq -c^2, -c^2 \geq \eta \geq -a^2$ . The coordinate surfaces of constant  $\xi$  and  $\eta$  become respectively confocal oblate spheroids and confocal hyperboloids of revolution of one sheet (Fig. 8). As the third coordinate we can take the polar angle  $\phi$  in the  $xy$ -plane ( $x = \rho \cos \phi, y = \rho \sin \phi$ ). For  $a = b$  the ellipsoidal coordinate  $\zeta$  degenerates to a constant,  $-a^2$ . Its relation to the angle  $\phi$  is given by the way in which it tends to  $-a^2$  as  $b$  tends to  $a$ , namely

$$\cos \phi = \sqrt{[(a^2 + \zeta)/(a^2 - b^2)]} \quad \text{as } b \rightarrow a. \quad (4.8)$$

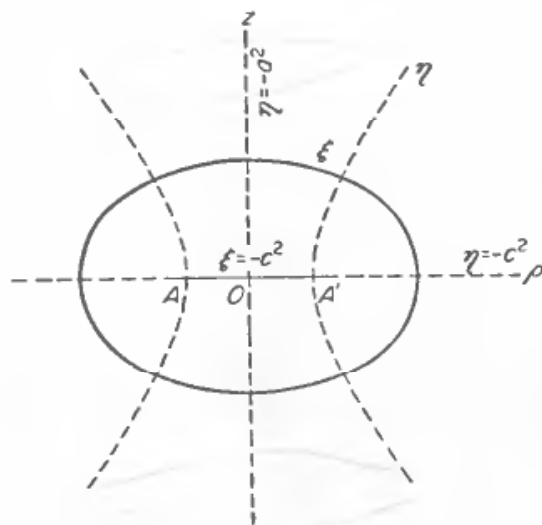


FIG. 8

This is easily seen from (4.4) or directly from (4.1). The relation between the coordinates  $z, \rho$  and  $\xi, \eta$  is given, according to (4.4), by

$$z = \pm \sqrt{\left[ \frac{(\xi + c^2)(\eta + c^2)}{c^2 - a^2} \right]}, \quad \rho = \sqrt{\left[ \frac{(\xi + a^2)(\eta + a^2)}{a^2 - c^2} \right]}. \quad (4.9)$$

The coordinates  $\xi, \eta, \phi$  are called *oblate spheroidal coordinates*.†

Similarly, for  $a > b = c$  ellipsoidal coordinates become *prolate spheroidal coordinates*. Two coordinates  $\xi$  and  $\zeta$  are roots of the equation

$$\frac{x^2}{a^2 + u} + \frac{\rho^2}{b^2 + u} = 1, \quad \rho^2 = y^2 + z^2, \quad (4.10)$$

where  $\xi \geq -b^2, -b^2 \geq \zeta \geq -a^2$ . The surfaces of constant  $\xi$  and  $\zeta$  are prolate spheroids

† We here define spheroidal coordinates to be the limit of ellipsoidal coordinates. Other definitions are used in the literature, but are easily related to ours.

and hyperboloids of revolution of two sheets (Fig. 9). The coordinate  $\eta$  degenerates to a constant,  $-b^2$ , for  $c \rightarrow b$ , and we have

$$\cos \phi = \sqrt{[(b^2 + \eta)/(b^2 - c^2)]}, \quad (4.11)$$

where  $\phi$  is the polar angle in the  $yz$ -plane. The relation between the coordinates  $x, \rho$  and  $\xi, \zeta$  is given by

$$x = \pm \sqrt{\left[ \frac{(\xi + a^2)(\zeta + a^2)}{a^2 - b^2} \right]}, \quad \rho = \sqrt{\left[ \frac{(\xi + b^2)(\zeta + b^2)}{b^2 - a^2} \right]}. \quad (4.12)$$

In a system of oblate spheroidal coordinates the foci of the spheroids and hyperboloids lie on a circle of radius  $\sqrt{(a^2 - c^2)}$  in the  $xy$ -plane; in Fig. 8  $AA'$  is a diameter of this circle. Let us draw a plane passing through the  $z$ -axis and some point  $P$ . It intersects the focal circle at two points; let their distances from  $P$  be  $r_1, r_2$ . If the coordinates of  $P$  are  $\rho, z$ , then

$$r_1^2 = [\rho - \sqrt{(a^2 - c^2)}]^2 + z^2, \quad r_2^2 = [\rho + \sqrt{(a^2 - c^2)}]^2 + z^2.$$

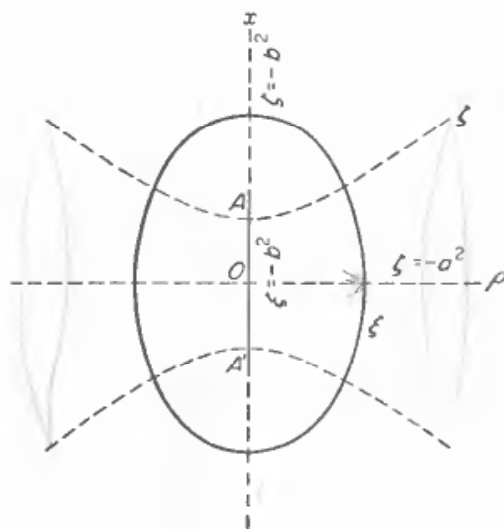


FIG. 9

The spheroidal coordinates  $\xi, \eta$  are given in terms of  $r_1, r_2$  by

$$\xi = \frac{1}{4}(r_1 + r_2)^2 - a^2, \quad \eta = \frac{1}{4}(r_2 - r_1)^2 - a^2. \quad (4.13)$$

In a system of prolate spheroidal coordinates the foci are the points  $x = \pm \sqrt{(a^2 - b^2)}$  on the  $x$ -axis (the points  $A, A'$  in Fig. 9). If  $r_1$  and  $r_2$  are the distances of these foci from  $P$ , then

$$r_1^2 = \rho^2 + [x - \sqrt{(a^2 - b^2)}]^2, \quad r_2^2 = \rho^2 + [x + \sqrt{(a^2 - b^2)}]^2,$$

and the spheroidal coordinates  $\xi, \zeta$  are given in terms of  $r_1, r_2$  by the same formulae (4.13), with  $\zeta$  in place of  $\eta$ .

Let us now turn to the problem of the field of a charged ellipsoid whose surface is given by the equation (4.3). In ellipsoidal coordinates this is the surface  $\xi = 0$ . It is therefore clear that, if we seek the field potential as a function of  $\xi$  only, all the ellipsoidal surfaces  $\xi = \text{constant}$ , and in particular the surface of the conductor, will be equipotential surfaces.

Laplace's equation (4.6) then becomes

$$\frac{d}{d\xi} \left( R_\xi \frac{d\phi}{d\xi} \right) = 0,$$

whence

$$\phi(\xi) = A \int_{\xi}^{\infty} \frac{d\xi}{R_\xi}.$$

The upper limit of integration is taken so that the field is zero at infinity. The constant  $A$  is most simply determined from the condition that at large distances  $r$  the field must become a Coulomb field and  $\phi \cong e/r$ , where  $e$  is the total charge on the conductor. When  $r \rightarrow \infty$ ,  $\xi \rightarrow \infty$ , and  $\xi \cong r^2$ , as we see from equation (4.1) with  $u = \xi$ . For large  $\xi$  we have  $R_\xi \cong \xi^{3/2}$ , and  $\phi \cong 2A/\sqrt{\xi} = 2A/r$ . Hence  $2A = e$ , and therefore

$$\phi(\xi) = \frac{1}{2}e \int_{\xi}^{\infty} \frac{d\xi}{R_\xi}. \quad (4.14)$$

The integral is an elliptic integral of the first kind. The surface of the conductor corresponds to  $\xi = 0$ , and so the capacitance of the conductor is given by

$$\frac{1}{C} = \frac{1}{2} \int_0^{\infty} \frac{d\xi}{R_\xi} \quad (4.15)$$

The distribution of charge on the surface of the ellipsoid is determined by the normal derivative of the potential:

$$\sigma = -\frac{1}{4\pi} \left[ \frac{\partial \phi}{\partial n} \right]_{\xi=0} = -\frac{1}{4\pi} \left[ \frac{1}{h_1} \frac{d\phi}{d\xi} \right]_{\xi=0} = \frac{e}{4\pi} \frac{1}{\sqrt{(\eta\xi)}}.$$

From equations (4.4) we easily see that for  $\xi = 0$

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} = \frac{\eta\xi}{a^2b^2c^2}.$$

Hence

$$\sigma = \frac{e}{4\pi abc} \left( \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right)^{-\frac{1}{2}} \quad (4.16)$$

For a spheroid the integrals (4.14), (4.15) degenerate and can be expressed in terms of elementary functions. For a prolate spheroid ( $a > b = c$ ) the field potential is

$$\phi = \frac{e}{\sqrt{(a^2 - b^2)}} \tanh^{-1} \sqrt{\frac{a^2 - b^2}{\xi + a^2}}, \quad (4.17)$$

and the capacitance is

$$C = \frac{\sqrt{(a^2 - b^2)}}{\cosh^{-1}(a/b)}. \quad (4.18)$$



For an oblate spheroid ( $a = b > c$ ) we have

$$\phi = \frac{e}{\sqrt{(a^2 - c^2)}} \tan^{-1} \sqrt{\frac{a^2 - c^2}{\xi + c^2}}, \quad C = \frac{\sqrt{(a^2 - c^2)}}{\cos^{-1}(c/a)}. \quad (4.19)$$

In particular, for a circular disc ( $a = b, c = 0$ )

$$C = 2a/\pi. \quad (4.20)$$

Let us now consider the problem of an uncharged conducting ellipsoid in a uniform external electric field  $\mathcal{E}$ . Without loss of generality we may take the field  $\mathcal{E}$  to be along one of the axes of the ellipsoid. In any other case this field may be resolved into components along the three axes, and the resultant field is a superposition of those arising from each component separately.

The potential of a uniform field  $\mathcal{E}$  along the  $x$ -axis (the  $a$ -axis of the ellipsoid) is, in ellipsoidal coordinates,

$$\phi_0 = -\mathcal{E}x = -\mathcal{E}\sqrt{[(\xi + a^2)(\eta + a^2)(\zeta + a^2)/(b^2 - a^2)(c^2 - a^2)]}. \quad (4.21)$$

We write the field potential outside the ellipsoid as  $\phi = \phi_0 + \phi'$ , where  $\phi'$  gives the required perturbation of the external field by the ellipsoid, and seek  $\phi'$  in the form

$$\phi' = \phi_0 F(\xi). \quad (4.22)$$

In this function the factors depending on  $\eta$  and  $\zeta$  are the same as in  $\phi_0$ ; this enables us to satisfy the boundary condition at  $\xi = 0$  for arbitrary  $\eta, \zeta$  (i.e. on the surface of the ellipsoid). Substituting (4.22) in Laplace's equation (4.6), we obtain for  $F(\xi)$  the equation

$$\frac{d^2 F}{d\xi^2} + \frac{dF}{d\xi} \frac{d}{d\xi} \log [R_\xi(\xi + a^2)] = 0.$$

One solution of this equation is  $F = \text{constant}$ , and the other is

$$F(\xi) = A \int_{\xi}^{\infty} \frac{d\xi}{(\xi + a^2)R_\xi}. \quad (4.23)$$

The upper limit of integration is taken so that  $\phi' \rightarrow 0$  for  $\xi \rightarrow \infty$ . The integral is an elliptic integral of the second kind.

We must have  $\phi = \text{constant}$  on the surface of the ellipsoid. For this condition to be satisfied with  $\xi = 0$  and arbitrary  $\eta, \zeta$ , the constant value of  $\phi$  must be zero. Determining the coefficient  $A$  in  $F(\xi)$  so that  $F(0) = -1$ , we obtain the following final expression for the field potential near the ellipsoid:

$$\phi = \phi_0 \left\{ 1 - \int_{\xi}^{\infty} \frac{ds}{(s + a^2)R_s} \bigg/ \int_0^{\infty} \frac{ds}{(s + a^2)R_s} \right\}. \quad (4.24)$$

Let us find the form of the potential  $\phi'$  at large distances  $r$  from the ellipsoid. For large  $r$ , the coordinate  $\xi$  is large, and  $\xi \cong r^2$ , as follows at once from equation (4.1). Hence

$$\int_{\xi}^{\infty} \frac{ds}{(s + a^2)R_s} \cong \int_{r^2}^{\infty} \frac{ds}{s^{5/2}} = \frac{2}{3r^3},$$

and the potential  $\phi' = \mathfrak{E}_x V / 4\pi n^{(x)} r^3$ , where  $V = \frac{4}{3}\pi abc$  is the volume of the ellipsoid and  $n^{(x)}, n^{(y)}, n^{(z)}$  are defined by

$$\left. \begin{aligned} n^{(x)} &= \frac{1}{2}abc \int_0^\infty \frac{ds}{(s+a^2)R_s}, & n^{(y)} &= \frac{1}{2}abc \int_0^\infty \frac{ds}{(s+b^2)R_s}, \\ n^{(z)} &= \frac{1}{2}abc \int_0^\infty \frac{ds}{(s+c^2)R_s}. \end{aligned} \right\} \quad (4.25)$$

The expression for  $\phi'$  is, as we should expect, the potential of an electric dipole:  $\phi' = x \mathcal{P}_x / r^3$ , where the dipole moment of the ellipsoid is

$$\mathcal{P}_x = \mathfrak{E}_x V / 4\pi n^{(x)}. \quad (4.26)$$

Analogous expressions give the dipole moment when the field  $\mathfrak{E}$  is along the  $y$  or  $z$  axis.

The positive constants  $n^{(x)}, n^{(y)}, n^{(z)}$  depend only on the shape of the ellipsoid, and not on its volume; they are called the *depolarizing factors*.† If the coordinate axes do not necessarily coincide with those of the ellipsoid, formula (4.26) must be written in the tensor form

$$(4\pi/V)n_{ik}\mathcal{P}_k = \mathfrak{E}_i. \quad (4.27)$$

The quantities  $n^{(x)}, n^{(y)}, n^{(z)}$  are the principal values of the symmetrical tensor  $n_{ik}$  of rank two. Comparison with the definition (2.13) shows that  $\alpha_{ik} = n^{-1}_{ik}/4\pi$  is the polarizability tensor of the conducting ellipsoid.

In the general case of arbitrary  $a, b, c$ , it follows from the definitions of  $n^{(x)}, n^{(y)}, n^{(z)}$  that

$$n^{(x)} < n^{(y)} < n^{(z)} \quad \text{if} \quad a > b > c. \quad (4.28)$$

Further, by adding the integrals for  $n^{(x)}, n^{(y)}, n^{(z)}$  and using as the variable of integration  $u = R_s^2$ , we find

$$n^{(x)} + n^{(y)} + n^{(z)} = \frac{1}{2}abc \int_{(abc)^2}^\infty \frac{du}{u^{3/2}},$$

whence

$$n^{(x)} + n^{(y)} + n^{(z)} = 1. \quad (4.29)$$

The sum of the three depolarizing factors is thus unity; in tensor notation,  $n_{ii} = 1$ . Since these coefficients are positive, none can exceed unity.

For a sphere ( $a = b = c$ ) it is evident from symmetry that

$$n^{(x)} = n^{(y)} = n^{(z)} = \frac{1}{3}. \quad (4.30)$$

For a cylinder with its axis in the  $x$ -direction ( $a \rightarrow \infty$ ), we have‡

$$n^{(x)} = 0, \quad n^{(y)} = n^{(z)} = \frac{1}{2}. \quad (4.31)$$

† The same coefficients occur in problems concerning a dielectric ellipsoid in an external electric field, or a magnetic ellipsoid in a magnetic field (§8). Tables and graphs of the coefficients for spheroids and ellipsoids have been given by E. C. Stoner (*Philosophical Magazine* [7] 36, 803, 1945) and J. A. Osborn (*Physical Review* 67, 351, 1945).

‡ These values for a sphere and a cylinder agree, of course, with those found in §3, Problems 1 and 2.

The limiting case  $a, b \rightarrow \infty$  (a flat plate) corresponds to the obvious values

$$n^{(x)} = n^{(y)} = 0, \quad n^{(z)} = 1.$$

The elliptic integrals (4.25) can be expressed in terms of elementary functions if the ellipsoid is a spheroid. For a prolate spheroid ( $a > b = c$ ) of eccentricity  $e = \sqrt{1 - b^2/a^2}$ ,

$$n^{(x)} = \frac{1 - e^2}{2e^3} \left( \log \frac{1 + e}{1 - e} - 2e \right), \quad n^{(y)} = n^{(z)} = \frac{1}{2}(1 - n^{(x)}). \quad (4.32)$$

If the spheroid is nearly spherical ( $e \ll 1$ ) we have approximately

$$n^{(x)} = \frac{1}{3} - \frac{2}{15}e^2, \quad n^{(y)} = n^{(z)} = \frac{1}{3} + \frac{1}{15}e^2. \quad (4.33)$$

For an oblate spheroid ( $a = b > c$ )

$$n^{(z)} = \frac{1 + e^2}{e^3} (e - \tan^{-1} e), \quad n^{(x)} = n^{(y)} = \frac{1}{2}(1 - n^{(z)}), \quad (4.34)$$

where  $e = \sqrt{a^2/c^2 - 1}$ . If  $e \ll 1$ , then

$$n^{(z)} = \frac{1}{3} + \frac{2}{15}e^2, \quad n^{(x)} = n^{(y)} = \frac{1}{3} - \frac{1}{15}e^2. \quad (4.35)$$

## PROBLEMS

**PROBLEM 1.** Find the field of a charged conducting circular disc with radius  $a$ , expressing it in cylindrical coordinates. Find the distribution of charge on the disc.

**SOLUTION.** The charge distribution is obtained by taking the limit of formula (4.16) as  $c \rightarrow 0$ ,  $z \rightarrow 0$ , with  $z/c = \sqrt{1 - r^2/a^2}$  (where  $r^2 = x^2 + y^2$ ), in accordance with (4.3). This gives

$$\sigma = \frac{e}{4\pi a^2} \left( 1 - \frac{r^2}{a^2} \right)^{-\frac{1}{2}}$$

The field potential is given in all space by formula (4.19), where we put  $c = 0$  and express  $\xi$  in terms of  $r$  and  $z$  by means of equation (4.1) with  $c = 0$ ,  $u = \xi$ ,  $a = b$ :

$$\phi = \frac{e}{a} \tan^{-1} \left[ \frac{2a^2}{r^2 + z^2 - a^2 + \sqrt{[(r^2 + z^2 - a^2)^2 + 4a^2 z^2]}} \right]^{\frac{1}{2}}.$$

Near the edge of the disc, we replace  $r$  and  $z$  by coordinates  $\rho$  and  $\theta$  such that  $z = \rho \sin \theta$ ,  $r = a - \rho \cos \theta$  (Fig. 10, p. 26;  $\rho \ll a$ ), obtaining

$$\phi \cong \frac{e}{a} \left( \frac{1}{2}\pi - \sqrt{\frac{2\rho}{a}} \sin \frac{1}{2}\theta \right),$$

in agreement with the general result derived in §3, Problem 3.

**PROBLEM 2.** Determine the electric quadrupole moment of a charged ellipsoid.

**SOLUTION.** The quadrupole moment tensor of a charged conductor is defined as  $D_{ik} = e(3\overline{x_i x_k} - \overline{r^2} \delta_{ik})$ , where  $e$  is the total charge, and the bar denotes an average such as

$$\overline{x_i x_k} = \frac{1}{e} \oint x_i x_k \sigma \, df.$$

It is evident that the axes of the ellipsoid are also the principal axes of the tensor  $D_{ik}$ . Using formula (4.16) for  $\sigma$ , and for the element of surface of the ellipsoid the expression

$$df = \frac{dx \, dy}{v_z} = \frac{dx \, dy}{z/c^2} \sqrt{\left[ \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right]},$$

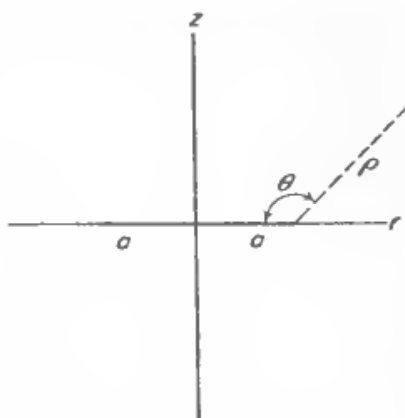


FIG. 10

where  $\mathbf{v}$  is a unit vector normal to the surface, we obtain

$$\overline{z^2} = \frac{c}{4\pi ab} \int z \, dx \, dy = \frac{1}{3}c^2;$$

the integration over  $x$  and  $y$  covers twice the area of the cross-section of the ellipsoid by the  $xy$ -plane. Thus

$$D_{xx} = \frac{1}{3}e(2a^2 - b^2 - c^2), \quad D_{yy} = \frac{1}{3}e(2b^2 - c^2 - a^2), \quad D_{zz} = \frac{1}{3}e(2c^2 - a^2 - b^2).$$

**PROBLEM 3.** Determine the distribution of charge on the surface of an uncharged conducting ellipsoid placed in a uniform external field.

**SOLUTION.** According to formula (1.9) we have

$$\sigma = -\frac{1}{4\pi} \left[ \frac{\partial \phi}{\partial n} \right]_{\xi=0} = - \left[ \frac{1}{4\pi h_1} \frac{\partial \phi}{\partial \xi} \right]_{\xi=0};$$

by (4.5) the element of length along the normal to the surface of the ellipsoid is  $h_1 d\xi$ . Substituting (4.24) and using the fact that

$$v_x = \left[ \frac{1}{h_1} \frac{\partial x}{\partial \xi} \right]_{\xi=0} = \left[ \frac{x}{2a^2 h_1} \right]_{\xi=0},$$

we have  $\sigma = \mathcal{E} v_x / 4\pi n^{(x)}$  when the external field is in the  $x$ -direction. When the direction of the external field is arbitrary this becomes

$$\sigma = \frac{1}{4\pi} v_i n^{-1}_{ik} \mathcal{E}_k = \frac{1}{4\pi} \left[ \frac{v_x}{n^{(x)}} \mathcal{E}_x + \frac{v_y}{n^{(y)}} \mathcal{E}_y + \frac{v_z}{n^{(z)}} \mathcal{E}_z \right].$$

**PROBLEM 4.** The same as Problem 3, but for a plane circular disc with radius  $a$  lying parallel to the field.† Determine also the dipole moment of the disc.

**SOLUTION.** Let us regard the disc as the limit of a spheroid when the semiaxis  $c$  tends to zero. The depolarizing factor along this axis (the  $z$ -axis) tends to 1, and those along the  $x$  and  $y$  axes tend to zero:  $n^{(z)} = 1 - \pi c / 2a$ ,  $n^{(x)} = n^{(y)} = \pi c / 4a$ , by (4.34). The component  $v_x$  of the unit vector along the normal to the surface of the spheroid tends to zero:

$$v_x = \frac{x}{a^2} \left( \frac{x^2 + y^2}{a^4} + \frac{z^2}{c^4} \right)^{-\frac{1}{2}} \rightarrow \frac{x}{a^2} \frac{c^2}{z} = \frac{xc}{a^2} \left( 1 - \frac{x^2 + y^2}{a^2} \right)^{-\frac{1}{2}}.$$

Hence the charge density is

$$\sigma = \frac{\mathcal{E} v_x}{4\pi n^{(x)}} = \frac{\mathcal{E} \rho \cos \phi}{\pi^2 \sqrt{(a^2 - \rho^2)}},$$

where  $\rho$  and  $\phi$  are polar coordinates in the plane of the disc.

† The problem for a disc lying perpendicular to the field is trivial: the field remains uniform in all space, and charges  $\sigma = \pm \mathcal{E} / 4\pi$  are induced on the two sides of the disc.

The dipole moment of the disc is obtained from formula (4.26), and is  $\mathcal{P} = 4a^3\mathcal{E}/3\pi$ . Thus it is proportional to  $a^3$ , and not to the "volume"  $a^2c$  of the disc.

**PROBLEM 5.** Determine the field potential outside an uncharged conducting spheroid with its axis of symmetry parallel to a uniform external field.

**SOLUTION.** For a prolate spheroid ( $a > b = c$ , with the field  $\mathcal{E}$  in the  $x$ -direction) we find, on calculating the integral in formula (4.24),

$$\phi = -\mathcal{E}_x \left\{ 1 - \frac{\tanh^{-1} \sqrt{[(a^2 - b^2)/(\xi + a^2)]} - \sqrt{[(a^2 - b^2)/(\xi + a^2)]}}{\tanh^{-1} \sqrt{(1 - b^2/a^2)} - \sqrt{(1 - b^2/a^2)}} \right\}.$$

The coordinate  $\xi$  is related to  $x$  and  $\rho = \sqrt{(y^2 + z^2)}$  by

$$\frac{\rho^2}{b^2 + \xi} + \frac{x^2}{a^2 + \xi} = 1,$$

with  $0 \leq \xi \leq \infty$  in the space outside the ellipsoid.

For an oblate spheroid ( $a = b > c$ ) the field  $\mathcal{E}$  is along the  $z$ -axis. We must therefore replace  $s + a^2$  by  $s + c^2$  and put  $\phi_0 = -\mathcal{E}_z$  in the integrals in (4.24). Then

$$\phi = -\mathcal{E}_z \left\{ 1 - \frac{\sqrt{[(a^2 - c^2)/(\xi + c^2)]} - \tan^{-1} \sqrt{[(a^2 - c^2)/(\xi + c^2)]}}{\sqrt{(a^2/c^2 - 1)} - \tan^{-1} \sqrt{(a^2/c^2 - 1)}} \right\},$$

where the coordinate  $\xi$  is related to  $z$  and  $\rho = \sqrt{(x^2 + y^2)}$  by

$$\frac{\rho^2}{a^2 + \xi} + \frac{z^2}{c^2 + \xi} = 1.$$

**PROBLEM 6.** The same as Problem 5, but with the axis of symmetry perpendicular to the external field.

**SOLUTION.** For a prolate spheroid (with the field along the  $z$ -axis)

$$\phi = -\mathcal{E}_z \left\{ 1 - \frac{\sqrt{(\xi + a^2)/(\xi + b^2)} - (a^2 - b^2)^{-\frac{1}{2}} \tanh^{-1} \sqrt{[(a^2 - b^2)/(\xi + a^2)]}}{a/b^2 - (a^2 - b^2)^{-\frac{1}{2}} \tanh^{-1} \sqrt{(1 - b^2/a^2)}} \right\}.$$

For an oblate spheroid (with the field along the  $x$ -axis)

$$\phi = -\mathcal{E}_x \left\{ 1 - \frac{(a^2 - c^2)^{-\frac{1}{2}} \tan^{-1} \sqrt{[(a^2 - c^2)/(\xi + c^2)]} - \sqrt{(\xi + c^2)/(\xi + a^2)}}{(a^2 - c^2)^{-\frac{1}{2}} \tan^{-1} \sqrt{(a^2/c^2 - 1)} - c/a^2} \right\}.$$

**PROBLEM 7.** A uniform field  $\mathcal{E}$  in the  $z$ -direction (in the half-space  $z < 0$ ) is bounded by an earthed conducting plane at  $z = 0$ , containing a circular aperture. Determine the field and charge distribution on the plane.

**SOLUTION.** The  $xy$ -plane with a circular aperture of radius  $a$  and centre at the origin may be regarded as the limit of the hyperboloids of revolution of one sheet

$$\frac{\rho^2}{a^2 - |\eta|} - \frac{z^2}{|\eta|} = 1, \quad \rho^2 = x^2 + y^2,$$

as  $|\eta| \rightarrow 0$ . These hyperboloids are one of the families of coordinate surfaces in a system of oblate spheroidal coordinates with  $c = 0$ . The Cartesian coordinate  $z$ , according to (4.9), is given in terms of  $\xi$  and  $\eta$  by  $z = \sqrt{(\xi|\eta|)/a}$ , and  $\sqrt{\xi}$  must be taken with the positive and negative sign in the upper and lower half-space respectively.

Let us seek a solution in the form  $\phi = -\mathcal{E}_z F(\xi)$ . For the function  $F(\xi)$  we obtain

$$F(\xi) = \text{constant} \times \int \frac{d\xi}{\xi^{\frac{3}{2}}(\xi + a^2)} = \text{constant} \times \left[ \frac{a}{\sqrt{\xi}} - \tan^{-1} \frac{a}{\sqrt{\xi}} \right];$$

the constant of integration is put equal to zero in accordance with the condition  $\phi = 0$  for  $z \rightarrow +\infty$ , i.e.  $\sqrt{\xi} \rightarrow +\infty$ . The inverse tangent of a negative quantity must be taken as  $\tan^{-1}(a/\sqrt{\xi}) = \pi - \tan^{-1}(a/\sqrt{\xi})$ , and not as  $-\tan^{-1}(a/\sqrt{\xi})$  since the potential would then be discontinuous at the aperture ( $\xi = 0$ ). The constant coefficient is chosen so that, for  $z \rightarrow -\infty$  (i.e. for  $\sqrt{\xi} \rightarrow -\infty$  and  $\tan^{-1}(a/\sqrt{\xi}) \rightarrow \pi$ ),  $\phi \rightarrow -\mathcal{E}_z$ , and so we finally

have

$$\phi = -\frac{\mathcal{E}z}{\pi} \left[ \tan^{-1} \frac{a}{\sqrt{\xi}} - \frac{a}{\sqrt{\xi}} \right] = -\frac{\mathcal{E}}{\pi} \sqrt{|\eta|} \left[ \frac{\sqrt{\xi}}{a} \tan^{-1} \frac{a}{\sqrt{\xi}} - 1 \right].$$

On the conducting plane  $\eta = 0$  and the potential is zero, as it should be.

At large distances  $r = \sqrt{(z^2 + \rho^2)}$  from the aperture we have  $\xi \cong r^2$ , and the potential (in the upper half-space) is

$$\phi \cong \frac{\mathcal{E}a^2}{3\pi} \frac{\sqrt{-\eta}}{\xi} = \mathcal{E}a^3/3\pi r^3,$$

i.e. we have a dipole field, the moment of the dipole being  $\mathcal{P} = \mathcal{E}a^3/3\pi$ .

The field decreases as  $1/r^3$ , and therefore the flux of the field through an infinitely remote surface (in the half-space  $z > 0$ ) is zero. This means that all the lines of force passing through the aperture reach the upper side of the conducting plane.

The distribution of charge on the conducting plane is given by

$$\sigma = \mp \frac{1}{4\pi} \left[ \frac{\partial \phi}{\partial z} \right]_{z=0} = \mp \frac{a}{4\pi\sqrt{\xi}} \frac{\partial \phi}{\partial \sqrt{-\eta}} = \pm \frac{\mathcal{E}}{4\pi^2} \left[ \tan^{-1} \frac{a}{\sqrt{\xi}} - \frac{a}{\sqrt{\xi}} \right],$$

where the upper and lower signs refer to the upper and lower sides of the plane respectively. According to the formula

$$\frac{\rho^2}{a^2 + \xi} + \frac{z^2}{\xi} = 1,$$

which relates  $\xi$  to  $\rho, z$ , we have  $\sqrt{\xi} = \pm \sqrt{(\rho^2 - a^2)}$  on the plane  $z = 0$ . Thus the charge distribution on the lower side of the conducting plane is given by the formula

$$\sigma = -\frac{\mathcal{E}}{4\pi^2} \left( \pi - \sin^{-1} \frac{a}{\rho} + \frac{a}{\sqrt{(\rho^2 - a^2)}} \right).$$

As  $\rho \rightarrow \infty$  we have  $\sigma = -\mathcal{E}/4\pi$ , as we should expect. On the upper side

$$\sigma = -\frac{\mathcal{E}}{4\pi^2} \left( \frac{a}{\sqrt{(\rho^2 - a^2)}} - \sin^{-1} \frac{a}{\rho} \right).$$

The total induced charge on the upper side of the plane is finite:

$$e' = \int_a^\infty \sigma \cdot 2\pi\rho \cdot d\rho = -\frac{1}{8}a^2\mathcal{E}.$$

**PROBLEM 8.** The same as Problem 7, but for a plane with a slit of width  $2b$ .

**SOLUTION.** The  $xy$ -plane with a slit along the  $x$ -axis may be regarded as the limit of the hyperbolic cylinders

$$\frac{y^2}{b^2 - |\eta|} - \frac{z^2}{|\eta|} = 1$$

as  $|\eta| \rightarrow 0$ . These hyperbolic cylinders are one of the families of coordinate surfaces in a system of ellipsoidal coordinates with  $a \rightarrow \infty, c \rightarrow 0$ . The Cartesian coordinate  $z = \sqrt{(\xi|\eta|)}/b$ .

As in Problem 7, we seek a solution in the form  $\phi = -\mathcal{E}zF(\xi)$ , obtaining for the function  $F$

$$F = \text{constant} \times \int \frac{d\xi}{\xi^{\frac{3}{2}} \sqrt{(\xi + b^2)}}.$$

Here the coefficient and the constant of integration are determined by the conditions that  $F = 0$  and 1 for  $z \rightarrow +\infty$  and  $-\infty$  respectively (i.e. for  $\sqrt{\xi} \rightarrow +\infty$  and  $-\infty$ ), and the final result is

$$\phi = \frac{\mathcal{E}}{2b} [\sqrt{(\xi + b^2)} \mp \sqrt{\xi}] \sqrt{|\eta|},$$

where we now take  $\sqrt{\xi}$  positive and the two signs  $\mp$  correspond to the regions  $z > 0$  and  $z < 0$ .

At large distances from the slit we have in the upper half-space  $\xi \cong y^2 + z^2 = r^2$ , and the potential is

$\phi \cong \frac{1}{4}b\mathcal{E}\sqrt{(|\eta|/\xi)} = \frac{1}{4}\mathcal{E}b^2z/r^2$ , i.e. the field of a two-dimensional dipole of moment  $\frac{1}{8}\mathcal{E}b^2$  per unit length of the slit (see the formula in §3, Problem 2).

The distribution of charge on the conducting plane is given by

$$\sigma = -\frac{\mathcal{E}}{8\pi} \left( \frac{|y|}{\sqrt{(y^2 - b^2)}} \mp 1 \right).$$

The total induced charge on the upper side of the plane, per unit length of the slit, is

$$e' = 2 \int_b^\infty \sigma dy = -\mathcal{E}b/4\pi.$$

Near the edge of the slit we can take  $\xi \rightarrow 0$  in the expression for  $\phi(\xi, \eta)$ , obtaining

$$\eta \cong -2b\rho \sin^2 \frac{1}{2}\theta,$$

where  $\rho$  and  $\theta$  are polar coordinates in the  $yz$ -plane, measured from the slit edge ( $y = b + \rho \cos \theta$ ,  $z = \rho \sin \theta$ ). Then

$$\phi \cong \mathcal{E}\sqrt{(\frac{1}{2}b\rho)} \sin \frac{1}{2}\theta,$$

in agreement with the result in §3, Problem 3, for the case  $\theta_0 \ll 1$ .

## §5. The forces on a conductor

In an electric field certain forces act on the surface of a conductor. These forces are easily calculated as follows.

The momentum flux density in an electric field in a vacuum is given by the *Maxwell stress tensor*:†

$$-\sigma_{ik} = \frac{1}{4\pi} (\frac{1}{2}E^2\delta_{ik} - E_iE_k).$$

The force on an element  $df$  of the surface of the body is just the flux of momentum through it from outside, and is therefore  $\sigma_{ik}df_k = \sigma_{ik}n_kdf$  (the sign is changed because the normal vector  $\mathbf{n}$  is outwards and not inwards). The quantity  $\sigma_{ik}n_k$  is thus the force  $\mathbf{F}_s$  per unit area of the surface. Since, at the surface of a conductor, the field  $\mathbf{E}$  has no tangential component, we obtain

$$\mathbf{F}_s = \mathbf{n}E^2/8\pi, \quad (5.1)$$

or, introducing the surface charge density  $\sigma$ ,

$$\mathbf{F}_s = 2\pi\sigma^2\mathbf{n} = \frac{1}{2}\sigma\mathbf{E}.$$

We therefore conclude that a “negative pressure” acts on the surface of a conductor; it is directed along the outward normal to the surface, and its magnitude is equal to the energy density in the field.

The total force  $\mathbf{F}$  on the conductor is obtained by integrating the force (5.1) over the whole surface‡:

$$\mathbf{F} = \oint (E^2/8\pi) d\mathbf{f}. \quad (5.2)$$

† See *Fields*, §33. The stress tensor  $\sigma_{ik}$  is usually defined with the opposite sign to the momentum flux density tensor. This definition has been used elsewhere in the *Course of Theoretical Physics*, but by an oversight the definition of  $\sigma_{ik}$  in *Fields*, §33, had the other sign.

‡ In the present case we are applying this formula to a surface which does not precisely coincide with that of the body, but is some distance away, in order to exclude the effect of the field structure near the surface (see §1).

Usually, however, it is more convenient to calculate this quantity from the general laws of mechanics, by differentiating the energy  $\mathcal{U}$ . The force, in the direction of a coordinate  $q$ , acting on a conductor is  $-\partial \mathcal{U} / \partial q$ , where the derivative signifies the rate of change of energy when the body is translated in the  $q$ -direction. The energy must be expressed in terms of the charges on the conductors (which give rise to the field), and the differentiation is performed with the charges constant. Denoting this by the suffix  $e$ , we write

$$F_q = -(\partial \mathcal{U} / \partial q)_e. \quad (5.3)$$

Similarly, the projection, on any axis, of the total moment of the forces on the conductor is

$$K = -(\partial \mathcal{U} / \partial \psi)_e, \quad (5.4)$$

where  $\psi$  is the angle of rotation of the body about that axis.

If, however, the energy is expressed as a function of the potentials of the conductors, and not of their charges, the calculation of the forces from the energy requires special consideration. The reason is that, to maintain constant the potential of a moving conductor, it is necessary to use other bodies. For example, the potential of a conductor can be kept constant by connecting it to another conductor of very large capacitance, a "charge reservoir". On receiving a charge  $e_a$ , the conductor takes it from the reservoir, whose potential  $\phi_a$  is unchanged on account of its large capacitance, although its energy is reduced by  $e_a \phi_a$ . When the whole system of conductors receives charges  $e_a$ , the energy of the reservoirs connected to them changes by a total of  $-\sum e_a \phi_a$ . Only the energy of the conductors, and not that of the reservoirs, appears in  $\mathcal{U}$ . In this sense we can say that  $\mathcal{U}$  pertains to a system which is not energetically closed. Thus, for a system of conductors whose potentials are kept constant, the part of the mechanical energy is played not by  $\mathcal{U}$ , but by

$$\tilde{\mathcal{U}} = \mathcal{U} - \sum_a e_a \phi_a. \quad (5.5)$$

Substituting (2.2), we find that  $\mathcal{U}$  and  $\tilde{\mathcal{U}}$  differ only in sign:

$$\tilde{\mathcal{U}} = -\mathcal{U}. \quad (5.6)$$

The force  $F_q$  is obtained by differentiating  $\tilde{\mathcal{U}}$  with respect to  $q$  for constant potentials, i.e.

$$F_q = -(\partial \tilde{\mathcal{U}} / \partial q)_\phi = (\partial \mathcal{U} / \partial q)_\phi. \quad (5.7)$$

Thus the forces acting on a conductor can be obtained by differentiating  $\mathcal{U}$  either for constant charges or for constant potentials, the only difference being that the derivative must be taken with the minus sign in the first case and with the plus sign in the second.

The same result could be obtained more formally by starting from the differential identity

$$d\mathcal{U} = \sum_a \phi_a de_a - F_q dq, \quad (5.8)$$

in which  $\mathcal{U}$  is regarded as a function of the charges on the conductors and the coordinate  $q$ . This identity states that  $\partial \mathcal{U} / \partial e_a = \phi_a$  and  $\partial \mathcal{U} / \partial q = -F_q$ . Using the variables  $\phi_a$  instead of  $e_a$ , we have

$$d\tilde{\mathcal{U}} = -\sum_a e_a d\phi_a - F_q dq, \quad (5.9)$$

which gives (5.7).



At the end of §2 we have discussed the energy of a conductor in a uniform external electric field. The total force on an uncharged conductor in a uniform field is, of course, zero. The expression for the energy (2.14) can, however, be used to determine the force acting on a conductor in a quasi-uniform field  $\mathcal{E}$ , i.e. a field which varies only slightly over the dimensions of the conductor. In such a field the energy can still be calculated, to a first approximation, from formula (2.14), and the force  $\mathbf{F}$  is the gradient of this energy:

$$\mathbf{F} = -\text{grad } \mathcal{U} = \frac{1}{2}\alpha_{ik} V \text{ grad } (\mathcal{E}_i \mathcal{E}_k). \quad (5.10)$$

The total torque  $\mathbf{K}$  is in general non-zero even in a uniform external field. By the general laws of mechanics  $\mathbf{K}$  can be determined by considering an infinitesimal virtual rotation of the body. The change in energy in such a rotation is related to  $\mathbf{K}$  by  $\delta \mathcal{U} = -\mathbf{K} \cdot \delta \psi$ ,  $\delta \psi$  being the angle of the rotation. A rotation through an angle  $\delta \psi$  in a uniform field is equivalent to a rotation of the field through an angle  $-\delta \psi$  relative to the body. The change in the field is  $\delta \mathcal{E} = -\delta \psi \times \mathcal{E}$ , and the change in energy is

$$\delta \mathcal{U} = (\partial \mathcal{U} / \partial \mathcal{E}) \cdot \delta \mathcal{E} = -\delta \psi \cdot \mathcal{E} \times \partial \mathcal{U} / \partial \mathcal{E}.$$

But  $\partial \mathcal{U} / \partial \mathcal{E} = -\mathcal{P}$ , as we see from a comparison of formulae (2.13) and (2.14). Hence  $\delta \mathcal{U} = -\mathcal{P} \times \mathcal{E} \cdot \delta \psi$ , whence

$$\mathbf{K} = \mathcal{P} \times \mathcal{E}, \quad (5.11)$$

in accordance with the usual expression given by the theory of fields in a vacuum.

If the total force and torque on a conductor are zero, the conductor remains at rest in the field, and effects involving the deformation of the body (called *electrostriction*) become important. The forces (5.1) on the surface of the conductor result in changes in its shape and volume. Because the force is an extending one, the volume of the body increases. A complete determination of the deformation requires a solution of the equations of the theory of elasticity, with the given distribution of forces (5.1) on the surface of the body. If, however, we are interested only in the change in volume, the problem can be solved very simply.

To do so, we must bear in mind that, if the deformation is slight (as in fact is true for electrostriction), the effect of the change of shape on the change of volume is of the second order of smallness. In the first approximation, therefore, the change in volume can be regarded as the result of deformation without change in shape, i.e. as a volume expansion under the action of some effective excess pressure  $\Delta P$  which is uniformly distributed over the surface of the body and replaces the exact distribution given by (5.1). The relative change in volume is obtained by multiplying  $\Delta P$  by the coefficient of uniform expansion of the substance. The pressure  $\Delta P$  is given, according to a well-known formula, by the derivative of the electric energy  $\mathcal{U}$  of the body with respect to its volume:  $\Delta P = -\partial \mathcal{U} / \partial V$ .†

Let the deforming field be due to the charged conductor itself. Then the energy  $\mathcal{U} = \frac{1}{2}e^2/C$ , and the pressure is  $\Delta P = -\frac{1}{2}e^2 \partial C^{-1} / \partial V$ . For a given shape, the capacitance of the body (having the dimensions of length) is proportional to the linear dimension, i.e. to  $V^{1/3}$ . Hence

$$\Delta P = e^2 / 6CV = e\phi / 6V. \quad (5.12)$$

† The quantity thus determined is the pressure exerted on the surface by the body itself; the pressure acting on the surface from outside is obtained by changing the sign.

If an uncharged conductor is situated in a uniform external field  $\mathcal{E}$ , its energy is given by formula (2.14). The extending pressure is therefore

$$\Delta P = \frac{1}{2} \alpha_{ik} \mathcal{E}_i \mathcal{E}_k. \quad (5.13)$$

## PROBLEMS

**PROBLEM 1.** A small conductor with capacitance  $c$  (equal in order of magnitude to its dimension) is at a distance  $r$  from the centre of a spherical conductor with large radius  $a$  ( $\gg c$ ). The distance  $r - a$  from the conductor to the surface of the sphere is supposed large compared with  $c$ , but not large compared with  $a$ . The two conductors are joined by a thin wire, so that they are at the same potential  $\phi$ . Determine the force of their mutual repulsion.

**SOLUTION.** Since the conductor  $c$  is small, we can suppose that its potential is the sum of the potential  $\phi a/r$  at a distance  $r$  from the centre of the large sphere and the potential  $e/c$  due to the charge  $e$  on the conductor itself. Hence  $\phi = \phi a/r + e/c$ , or  $e = c\phi(1 - a/r)$ . The required force of interaction  $F$  is the Coulomb repulsion between the charge  $e$  on the conductor and the charge  $a\phi$  on the sphere:

$$F = \frac{ac\phi^2}{r^2} \left(1 - \frac{a}{r}\right).$$

This expression is correct to within terms of higher order in  $c$ . The force is greatest when  $r = 3a/2$ , and its value there is  $F_{\max} = 4c\phi^2/27a$ , decreasing on either side of this distance.

**PROBLEM 2.** A charged conducting sphere is cut in half. Determine the force of repulsion between the hemispheres.†

**SOLUTION.** We imagine the hemispheres separated by an infinitely narrow slit, and determine the force  $F$  on each of them by integrating over the surface the force  $(E^2/8\pi) \cos \theta$ , which is the component of (5.1) in a direction perpendicular to the plane of separation of the hemispheres. In the slit  $E = 0$ , and on the outer surface  $E = e/a^2$ , where  $a$  is the radius of the sphere and  $e$  the total charge on it. The result is  $F = e^2/8a^2$ .

**PROBLEM 3.** The same as Problem 2, but for an uncharged sphere in a uniform external field  $\mathcal{E}$  perpendicular to the plane of separation.

**SOLUTION.** As in Problem 2, except that the field on the surface of the sphere is  $E = 3\mathcal{E} \cos \theta$  (§3, Problem 1). The required force is  $F = 9a^2\mathcal{E}^2/16$ .

**PROBLEM 4.** Determine the change in volume and in shape of a conducting sphere in a uniform external electric field.

**SOLUTION.** The change in volume  $\Delta V/V = \Delta P/K$ , where  $K$  is the modulus of volume expansion of the material, and  $\Delta P$  is given by formula (5.13). For a sphere,  $\alpha_{ik} = \delta_{ik}\alpha = 3\delta_{ik}/4\pi$  (§3, Problem 1), so that  $\Delta V/V = 3\mathcal{E}^2/8\pi K$ .

As a result of the deformation, the sphere is changed into a prolate spheroid. To determine the eccentricity, we may regard the deformation as a uniform pure shear in the volume of the body, just as, to determine the change in the total volume, we regarded it as a uniform volume expansion.

The condition of equilibrium for a deformed body may be formulated as requiring that the sum of the electrostatic and elastic energies should be a minimum. The former is, by (2.12) and (4.26),

$$\mathcal{U}_{es} = -\frac{V}{8\pi n} \mathcal{E}^2 \cong -\frac{3V\mathcal{E}^2}{8\pi} - \frac{3V}{10\pi} \frac{a-b}{R} \mathcal{E}^2,$$

where  $R$  is the original radius of the sphere,  $a$  and  $b$  the semiaxes of the spheroid, and  $n \cong \frac{1}{3} - 4(a-b)/15R$  is the depolarizing factor (see (4.33).)

Since the deformation is axially symmetrical about the direction of the field (the  $x$ -axis), only the components  $u_{xx}$  and  $u_{yy} = u_{zz}$  of the strain tensor are non-zero. Since we are considering equilibrium with respect to a change in shape, we can regard the volume as unchanged, i.e.  $u_{ii} = 0$ . Hence the elastic energy may be written

$$\mathcal{U}_{el} = \frac{1}{2} u_{ik} \sigma_{ik} V = \frac{1}{3} (\sigma_{xx} - \sigma_{yy}) (u_{xx} - u_{yy}) V,$$

† In Problems 2 and 3 we assume that the hemispheres are at the same potential.

where  $\sigma_{ik}$  is the elastic stress tensor (*TE*, §4). We have  $\sigma_{xx} - \sigma_{yy} = 2(u_{xx} - u_{yy})$ , where  $\mu$  is the modulus of rigidity of the material, and  $u_{xx} - u_{yy} = (a - b)/R$ . Hence

$$\mathcal{U}_{el} = \frac{2}{3}\mu(a - b)^2 V/R^2.$$

Making the sum  $\mathcal{U}_{es} + \mathcal{U}_{el}$  a minimum, we have  $(a - b)/R = 9\mathcal{E}^2/40\pi\mu$ .

**PROBLEM 5.** Find the relation between frequency and wavelength for waves propagated on a charged plane surface of a liquid conductor (in a gravitational field). Obtain the condition for this surface to be stable (Ya. I. Frenkel', 1935).

**SOLUTION.** Let the wave be propagated along the  $x$ -axis, with the  $z$ -axis vertically upwards. The vertical displacement of points on the surface of the liquid is  $\zeta = ae^{i(kx - \omega t)}$ . When the surface is at rest, the field above it is  $E_z = E = 4\pi\sigma_0$ , and its potential  $\phi = -4\pi\sigma_0 z$ , where  $\sigma_0$  is the surface charge density. The potential of the field above the oscillating surface can be written as  $\phi = -4\pi\sigma_0 z + \phi_1$ , with  $\phi_1 = \text{constant} \times e^{i(kx - \omega t)}e^{-kz}$ ,  $\phi_1$  being a small correction which satisfies the equation  $\Delta\phi_1 = 0$  and vanishes for  $z \rightarrow \infty$ . On the surface itself, the potential must have a constant value, which we take to be zero, and so  $\phi_1 = 4\pi\sigma_0\zeta$  for  $z = 0$ .

According to (5.1), an additional negative pressure acts on the charged surface of the liquid; this pressure is, as far as terms of the first order in  $\phi_1$ ,  $E^2/8\pi \cong E_z^2/8\pi \cong 2\pi\sigma_0^2 + [k\sigma_0\phi_1]_{z=0} = 2\pi\sigma_0^2 + 4\pi\sigma_0^2 k\zeta$ . The constant term  $2\pi\sigma_0^2$  is of no importance, since it can be included in the constant external pressure.

The consideration of the hydrodynamical motion in the wave is entirely analogous to the theory of capillary waves (*FM*, §61), differing only by the presence of the additional pressure mentioned above. At the surface of the liquid we have the boundary condition  $\rho g\zeta + \rho[\partial\Phi/\partial t]_{z=0} - \alpha\partial^2\zeta/\partial x^2 - 4\pi\sigma_0^2 k\zeta = 0$ , where  $\alpha$  is the surface-tension coefficient,  $\rho$  the density of the liquid, and  $\Phi$  its velocity potential.  $\Phi$  and  $\zeta$  are also related by  $\partial\zeta/\partial t = [\partial\Phi/\partial z]_{z=0}$ . Substituting in these two relations  $\zeta = ae^{i(kx - \omega t)}$  and  $\Phi = Ae^{i(kx - \omega t)}e^{-kz}$  ( $\Phi$  satisfies the equation  $\Delta\Phi = 0$ ) and eliminating  $a$  and  $A$ , we find the required relation between  $k$  and  $\omega$ :

$$\omega^2 = k(g\rho - 4\pi\sigma_0^2 k + \alpha k^2)/\rho. \quad (1)$$

If the surface of the liquid is to be stable, the frequency  $\omega$  must be real for all values of  $k$  (since otherwise there would be complex  $\omega$  with a positive imaginary part, and the factor  $e^{-i\omega t}$  would increase indefinitely). The condition for the right-hand side of (1) to be positive is  $(4\pi\sigma_0^2)^2 - 4g\rho\alpha < 0$ , or  $\sigma_0^4 < g\rho\alpha/4\pi^2$ . This is the condition for stability.

**PROBLEM 6.** Find the condition of stability for a charged spherical drop (Rayleigh, 1882).

**SOLUTION.** The sum of the electrostatic and surface energies of the drop is  $\mathcal{U} = e^2/2C + \alpha S$ , where  $\alpha$  is the surface-tension coefficient of the liquid,  $C$  the capacitance of the drop and  $S$  its surface area. Instability occurs (with increasing  $e$ ) with respect to deformation of the sphere into a spheroid, and does so when  $\mathcal{U}$  becomes a decreasing function of the eccentricity (for a given volume). The spherical shape always corresponds to an extremum of  $\mathcal{U}$ ; the stability condition is therefore  $[\partial^2\mathcal{U}/\partial(a-b)^2]_{a=b} > 0$ , where  $a$  and  $b$  are the semiaxes of the spheroid, and the differentiation is carried out with  $ab^2 = \text{constant}$ . Using the formula for the surface of a spheroid and (4.18) for its capacitance, we find after a somewhat lengthy calculation  $e^2 < 16\pi\alpha^3\alpha$ .

This condition ensures stability of the drop with respect to small deformations. It is found to be weaker than the condition for stability with respect to large deformations that divide the drop into two equal drops with charge  $\frac{1}{2}e$  and radius  $a/2^{1/3}$ :

$$e^2 < 16\pi\alpha^3\alpha(2^{1/3} - 1)/(2 - 2^{1/3}) = 0.35 \times 16\pi\alpha^3\alpha.$$

## CHAPTER II

# ELECTROSTATICS OF DIELECTRICS

### §6. The electric field in dielectrics

WE SHALL NOW go on to consider a static electric field in another class of substances, namely dielectrics. The fundamental property of dielectrics is that a steady current cannot flow in them. Hence the static electric field need not be zero, as in conductors, and we have to derive the equations which describe this field. One equation is obtained by averaging equation (1.3), and is again

$$\text{curl } \mathbf{E} = 0. \quad (6.1)$$

A second equation is obtained by averaging the equation  $\text{div } \mathbf{e} = 4\pi\rho$ :

$$\text{div } \mathbf{E} = 4\pi\bar{\rho}. \quad (6.2)$$

Let us suppose that no charges are brought into the dielectric from outside, which is the most usual and important case. Then the total charge in the volume of the dielectric is zero; even if it is placed in an electric field we have  $\int \bar{\rho} dV = 0$ . This integral equation, which must be valid for a body of any shape, means that the average charge density can be written as the divergence of a certain vector, which is usually denoted by  $-\mathbf{P}$ :

$$\bar{\rho} = -\text{div } \mathbf{P}, \quad (6.3)$$

while outside the body  $\mathbf{P} = 0$ . For, on integrating over the volume bounded by a surface which encloses the body but nowhere enters it, we find  $\int \bar{\rho} dV = -\int \text{div } \mathbf{P} dV = -\oint \mathbf{P} \cdot d\mathbf{f} = 0$ .  $\mathbf{P}$  is called the *dielectric polarization*, or simply the *polarization*, of the body. A dielectric in which  $\mathbf{P}$  differs from zero is said to be *polarized*. The vector  $\mathbf{P}$  determines not only the volume charge density (6.3), but also the density  $\sigma$  of the charges on the surface of the polarized dielectric. If we integrate formula (6.3) over an element of volume lying between two neighbouring unit areas, one on each side of the dielectric surface, we have, since  $\mathbf{P} = 0$  on the outer area (cf. the derivation of formula (1.9)),

$$\sigma = P_n \quad (6.4)$$

where  $P_n$  is the component of the vector  $\mathbf{P}$  along the outward normal to the surface.

To see the physical significance of the quantity  $\mathbf{P}$  itself, let us consider the total dipole moment of all the charges within the dielectric; unlike the total charge, the total dipole moment need not be zero. By definition, it is the integral  $\int \mathbf{r} \bar{\rho} dV$ . Substituting  $\bar{\rho}$  from (6.3) and again integrating over a volume which includes the whole body we have

$$\int \mathbf{r} \bar{\rho} dV = -\int \mathbf{r} \text{div } \mathbf{P} dV = -\oint \mathbf{r} (d\mathbf{f} \cdot \mathbf{P}) + \int (\mathbf{P} \cdot \text{grad}) \mathbf{r} dV.$$

The integral over the surface is zero, and in the second term we have  $(\mathbf{P} \cdot \text{grad}) \mathbf{r} = \mathbf{P}$ , so that

$$\int \mathbf{r} \bar{\rho} dV = \int \mathbf{P} dV. \quad (6.5)$$

Thus the polarization vector is the dipole moment (or *electric moment*) per unit volume of the dielectric.†

Substituting (6.3) in (6.2), we obtain the second equation of the electrostatic field in the form

$$\operatorname{div} \mathbf{D} = 0, \quad (6.6)$$

where we have introduced a quantity  $\mathbf{D}$  defined by

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \quad (6.7)$$

called the *electric induction*. The equation (6.6) has been derived by averaging the density of charges in the dielectric. If, however, charges not belonging to the dielectric are brought in from outside (we shall call these *extraneous charges*), then their density must be added to the right-hand side of equation (6.6):

$$\operatorname{div} \mathbf{D} = 4\pi\rho_{\text{ex}}. \quad (6.8)$$

On the surface of separation between two different dielectrics, certain boundary conditions must be satisfied. One of these follows from the equation  $\operatorname{curl} \mathbf{E} = 0$ . If the surface of separation is uniform as regards physical properties,‡ this condition requires the continuity of the tangential component of the field:

$$\mathbf{E}_{t1} = \mathbf{E}_{t2}; \quad (6.9)$$

cf. the derivation of the condition (1.7). The second condition follows from the equation  $\operatorname{div} \mathbf{D} = 0$ , and requires the continuity of the normal component of the induction:

$$D_{n1} = D_{n2}. \quad (6.10)$$

For a discontinuity in the normal component  $D_n = D_z$  would involve an infinity of the derivative  $\partial D_z / \partial z$ , and therefore of  $\operatorname{div} \mathbf{D}$ .

At a boundary between a dielectric and a conductor,  $\mathbf{E}_t = 0$ , and the condition on the normal component is obtained from (6.8):

$$\mathbf{E}_t = 0, \quad D_n = 4\pi\sigma_{\text{ex}}, \quad (6.11)$$

where  $\sigma_{\text{ex}}$  is the charge density on the surface of the conductor; cf. (1.8), (1.9).

## §7. The permittivity

In order that equations (6.1) and (6.6) should form a complete set of equations determining the electrostatic field, they must be supplemented by a relation between the induction  $\mathbf{D}$  and the field  $\mathbf{E}$ . In the great majority of cases this relation may be supposed linear. It corresponds to the first terms in an expansion of  $\mathbf{D}$  in powers of  $\mathbf{E}$ , and its correctness is due to the smallness of the external electric fields in comparison with the internal molecular fields.

The linear relation between  $\mathbf{D}$  and  $\mathbf{E}$  is especially simple in the most important case, that

† It should be noticed that the relation (6.3) inside the dielectric and the condition  $\mathbf{P} = 0$  outside do not in themselves determine  $\mathbf{P}$  uniquely; inside the dielectric we could add to  $\mathbf{P}$  any vector of the form  $\operatorname{curl} \mathbf{f}$ . The exact form of  $\mathbf{P}$  can be completely determined only by establishing its connection with the dipole moment.

‡ That is, as regards composition of the adjoining media, temperature, etc. If the dielectric is a crystal, the surface must be a crystallographic plane.

of an isotropic dielectric. It is evident that, in an isotropic dielectric, the vectors  $\mathbf{D}$  and  $\mathbf{E}$  must be in the same direction. The linear relation between them is therefore a simple proportionality:†

$$\mathbf{D} = \epsilon \mathbf{E}. \quad (7.1)$$

The coefficient  $\epsilon$  is the *permittivity* or *dielectric permeability* or *dielectric constant* of the substance and is a function of its thermodynamic state.

As well as the induction, the polarization also is proportional to the field:

$$\mathbf{P} = \kappa \mathbf{E} \equiv (\epsilon - 1)\mathbf{E}/4\pi. \quad (7.2)$$

The quantity  $\kappa$  is called the *polarization coefficient* of the substance, or its *dielectric susceptibility*. Later (§14) we shall show that the permittivity always exceeds unity; the polarizability, accordingly, is always positive. The polarizability of a rarefied medium (a gas) may be regarded as proportional to its density.

The boundary conditions (6.9) and (6.10) on the surface separating two isotropic dielectrics become

$$\mathbf{E}_{t1} = \mathbf{E}_{t2}, \quad \epsilon_1 E_{n1} = \epsilon_2 E_{n2}. \quad (7.3)$$

Thus the normal component of the field is discontinuous, changing in inverse proportion to the permittivity of the medium.

In a homogeneous dielectric,  $\epsilon = \text{constant}$ , and then it follows from  $\text{div } \mathbf{D} = 0$  that  $\text{div } \mathbf{P} = 0$ . By the definition (6.3) this means that the volume charge density in such a body is zero (but the surface density (6.4) is in general not zero). On the other hand, in an inhomogeneous dielectric we have a non-zero volume charge density

$$\bar{\rho} = -\text{div } \mathbf{P} = -\text{div} \frac{\epsilon - 1}{4\pi\epsilon} \mathbf{D} = -\frac{1}{4\pi} \mathbf{D} \cdot \text{grad} \frac{\epsilon - 1}{\epsilon} = -\frac{1}{4\pi\epsilon} \mathbf{E} \cdot \text{grad } \epsilon.$$

If we introduce the electric field potential by  $\mathbf{E} = -\text{grad } \phi$ , then equation (6.1) is automatically satisfied, and the equation  $\text{div } \mathbf{D} = \text{div } \epsilon \mathbf{E} = 0$  gives

$$\text{div } (\epsilon \text{ grad } \phi) = 0. \quad (7.4)$$

This equation becomes the ordinary Laplace's equation only in a homogeneous dielectric medium. The boundary conditions (7.3) can be rewritten as the following conditions on the potential:

$$\left. \begin{aligned} \phi_1 &= \phi_2, \\ \epsilon_1 \partial \phi_1 / \partial n &= \epsilon_2 \partial \phi_2 / \partial n; \end{aligned} \right\} \quad (7.5)$$

the continuity of the tangential derivatives of the potential is equivalent to the continuity of  $\phi$  itself.

In a dielectric medium which is piecewise homogeneous, equation (7.4) reduces in each homogeneous region to Laplace's equation  $\Delta \phi = 0$ , so that the permittivity appears in the solution of the problem only through the conditions (7.5). These conditions, however,

† This relation, which assumes that  $\mathbf{D}$  and  $\mathbf{E}$  vanish simultaneously, is, strictly speaking, valid only in dielectrics which are homogeneous as regards physical properties (composition, temperature, etc.). In inhomogeneous bodies  $\mathbf{D}$  may be non-zero even when  $\mathbf{E} = 0$ , and is determined by the gradients of thermodynamic quantities which vary through the body. The corresponding terms, however, are very small, and we shall use the relation (7.1) in what follows, even for inhomogeneous bodies.

involve only the ratio of the permittivities of two adjoining media. In particular, the solution of an electrostatic problem for a dielectric body with permittivity  $\epsilon_2$ , surrounded by a medium with permittivity  $\epsilon_1$ , is the same as for a body with permittivity  $\epsilon_2/\epsilon_1$ , surrounded by a vacuum.

Let us consider how the results obtained in Chapter I for the electrostatic field of conductors will be modified if these conductors are not in a vacuum but in a homogeneous and isotropic dielectric medium. In both cases the potential distribution satisfies the equation  $\Delta\phi = 0$ , with the boundary condition that  $\phi$  is constant on the surface of the conductor, and the only difference is that, instead of  $E_n = -\partial\phi/\partial n = 4\pi\sigma$ , we have

$$D_n = -\epsilon \partial\phi/\partial n = 4\pi\sigma, \quad (7.6)$$

giving the relation between the potential and the surface charge. Hence it is clear that the solution of the problem of the field of a charged conductor in a vacuum gives the solution of the same problem with a dielectric in place of the vacuum if we make the formal substitution  $\phi \rightarrow \epsilon\phi$ ,  $e \rightarrow e$  or  $\phi \rightarrow \phi$ ,  $e \rightarrow e/\epsilon$ . For given charges on the conductors, the potential and the field are reduced by a factor  $\epsilon$  in comparison with their values in a vacuum. This reduction in the field can be explained as the result of a partial "screening" of the charge on the conductor by the surface charges on the adjoining polarized dielectric. If, on the other hand, the potentials of the conductors are maintained, then the field is unchanged but the charges are increased by a factor  $\epsilon$ .†

Finally, it may be noted that in electrostatics we may formally regard a conductor (uncharged) as a body of infinite permittivity, in the sense that its effect on an external electric field is the same as that of a dielectric (of the same form) as  $\epsilon \rightarrow \infty$ . For, since the boundary condition on the induction  $\mathbf{D}$  is finite,  $\mathbf{D}$  must remain finite in the body even for  $\epsilon \rightarrow \infty$ . This means that  $\mathbf{E} \rightarrow 0$ , in accordance with the properties of conductors.

## PROBLEMS

**PROBLEM 1.** Determine the field due to a point charge  $e$  at a distance  $h$  from a plane boundary separating two different dielectric media.

**SOLUTION.** Let  $O$  be the position of the charge  $e$  in medium 1, and  $O'$  its image in the plane of separation, situated in medium 2 (Fig. 11, p. 38). We shall seek the field in medium 1 in the form of the field of two point charges,  $e$  and a fictitious charge  $e'$  at  $O'$  (cf. the method of images, §3):  $\phi_1 = e/\epsilon_1 r + e'/\epsilon_1 r'$ , where  $r$  and  $r'$  are the distances from  $O$  and  $O'$  respectively. In medium 2 we seek the field as that of a fictitious charge  $e''$  at  $O$ :  $\phi_2 = e''/\epsilon_2 r$ . On the boundary plane ( $r = r'$ ) the conditions (7.5) must hold, leading to the equations  $e - e' = e''$ ,  $(e + e')/\epsilon_1 = e''/\epsilon_2$ , whence

$$e' = e(\epsilon_1 - \epsilon_2)/(\epsilon_1 + \epsilon_2), \quad e'' = 2\epsilon_2 e/(\epsilon_1 + \epsilon_2). \quad (1)$$

For  $\epsilon_2 \rightarrow \infty$  we have  $e' = -e$ ,  $\phi_2 = 0$ , i.e. the result obtained in §3 for the field of a point charge near a conducting plane.

The force acting on the charge  $e$  (the *image force*) is

$$F = \frac{ee'}{(2h)^2\epsilon_1} = \left(\frac{e}{2h}\right)^2 \frac{\epsilon_1 - \epsilon_2}{\epsilon_1(\epsilon_1 + \epsilon_2)};$$

$F > 0$  corresponds to repulsion.

**PROBLEM 2.** The same as Problem 1, but for an infinite charged straight wire parallel to a plane boundary surface at a distance  $h$ .

† From this it follows, in particular, that when a capacitor is filled with a dielectric its capacitance increases by a factor  $\epsilon$ .



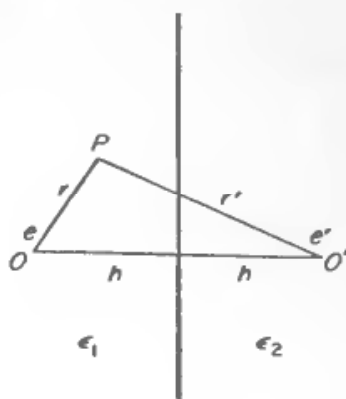


FIG. 11

**SOLUTION.** As in Problem 1, except that the field potentials in the two media are  $\phi_1 = -(2e/\epsilon_1) \log r - (2e'/\epsilon_1) \log r'$ ,  $\phi_2 = -(2e''/\epsilon_2) \log r$ , where  $e, e', e''$  are the charges per unit length of the wire and of its images, and  $r, r'$  are the distances in a plane perpendicular to the wire. The same expressions (1) are obtained for  $e', e''$ , and the force on unit length of the wire is  $F = 2ee'/2h\epsilon_1 = e^2(\epsilon_1 - \epsilon_2)/h\epsilon_1(\epsilon_1 + \epsilon_2)$ .

**PROBLEM 3.** Determine the field due to an infinite charged straight wire in a medium with permittivity  $\epsilon_1$ , lying parallel to a cylinder with radius  $a$  and permittivity  $\epsilon_2$ , at a distance  $b (> a)$  from its axis.†

**SOLUTION.** We seek the field in medium 1 as that produced in a homogeneous dielectric (with  $\epsilon_1$ ) by the actual wire (passing through  $O$  in Fig. 12), with charge  $e$  per unit length, and two fictitious wires with charges  $e'$  and  $-e'$  per unit length, passing through  $A$  and  $O'$  respectively. The point  $A$  is at a distance  $a^2/b$  from the axis of the cylinder. Then, for all points on the circumference, the distances  $r$  and  $r'$  from  $O$  and  $A$  are in a constant ratio  $r'/r = a/b$ , and so it is possible to satisfy the boundary conditions on this circumference. In medium 2 we seek the field as that produced in a homogeneous medium (with  $\epsilon_2$ ) by a fictitious charge  $e''$  on the wire passing through  $O$ .

The boundary conditions on the surface of separation are conveniently formulated in terms of the potential  $\phi$  ( $\mathbf{E} = -\text{grad } \phi$ ) and the vector potential  $\mathbf{A}$  (cf. §3), defined by  $\mathbf{D} = \text{curl } \mathbf{A}$  (in accordance with the equation  $\text{div } \mathbf{D} = 0$ ). In a two-dimensional problem,  $\mathbf{A}$  is in the  $z$ -direction (perpendicular to the plane of the figure). The conditions of continuity for the tangential components of  $\mathbf{E}$  and the normal component of  $\mathbf{D}$  are equivalent to  $\phi_1 = \phi_2$ ,  $A_1 = A_2$ .

For the field of a charged wire we have in polar coordinates  $r, \theta$  the equation  $\phi = -(2e/\epsilon) \log r + \text{constant}$ ,  $A = 2e\theta + \text{constant}$ ; cf. (3.18). Hence the boundary conditions are

$$\begin{aligned} \frac{2}{\epsilon_1}(-e \log r - e' \log r' + e' \log a) &= -\frac{2e''}{\epsilon_2} \log r + \text{constant}, \\ 2[e\theta + e'\theta' - e'(\theta + \theta')] &= 2e''\theta, \end{aligned}$$

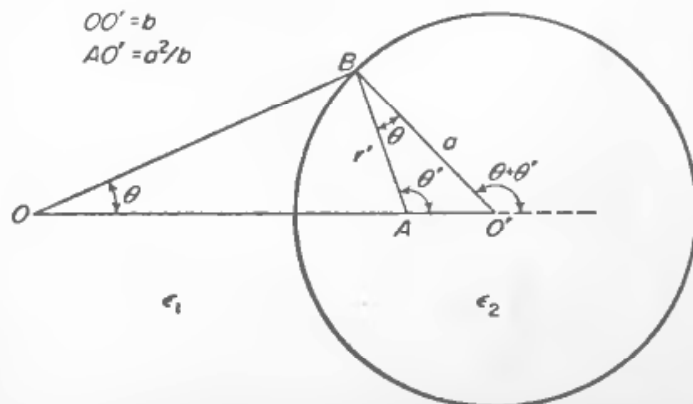


FIG. 12

† The corresponding problem of a point charge near a dielectric sphere cannot be solved in closed form.



where the angles are as shown in Fig. 12, and we have used the fact that  $OO'B$  and  $BO'A$  are similar triangles. Hence  $\epsilon_2(e + e') = \epsilon_1 e''$ ,  $e - e' = e''$ , and the expressions for  $e'$  and  $e''$  are again formulae (1) of Problem 1.

The force acting on unit length of the charged wire is parallel to  $OO'$ , and is

$$F = eE = \frac{2ee'}{\epsilon_1} \left( \frac{1}{OA} - \frac{1}{OO'} \right) = \frac{2e^2(\epsilon_1 - \epsilon_2)a^2}{\epsilon_1(\epsilon_1 + \epsilon_2)b(b^2 - a^2)},$$

$F > 0$  corresponds to repulsion. In the limit  $a, b \rightarrow \infty$ ,  $b - a \rightarrow h$ , this gives the result in Problem 1.

**PROBLEM 4.** The same as Problem 3, but for the case where the wire is inside a cylinder with permittivity  $\epsilon_2$  ( $b < a$ ).

**SOLUTION.** We seek the field in medium 2 as that due to the actual wire, with charge  $e$  per unit length ( $O$  in Fig. 13), and a fictitious wire with charge  $e'$  per unit length passing through  $A$ , which is now outside the cylinder. In medium 1 we seek the field as that of wires with charges  $e''$  and  $e - e''$  passing through  $O$  and  $O'$  respectively. By the same method as in the preceding problem we find  $e' = -e(\epsilon_1 - \epsilon_2)/(\epsilon_1 + \epsilon_2)$ ,  $e'' = 2\epsilon_1 e/(\epsilon_1 + \epsilon_2)$ . For  $\epsilon_2 > \epsilon_1$  the wire is repelled from the surface of the cylinder by a force

$$F = \frac{2ee'}{\epsilon_2} \frac{1}{OA} = \frac{2e^2(\epsilon_2 - \epsilon_1)b}{\epsilon_2(\epsilon_1 + \epsilon_2)(a^2 - b^2)}.$$

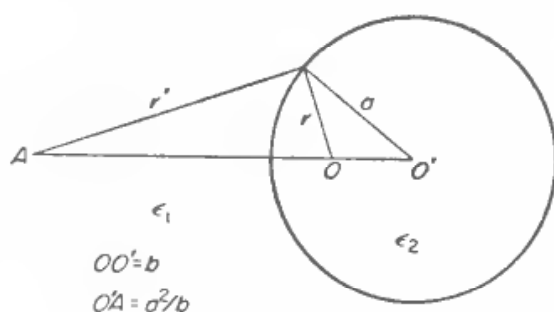


FIG. 13

**PROBLEM 5.** Show that the field potential  $\phi_A(\mathbf{r}_B)$  at a point  $\mathbf{r}_B$  in an arbitrary inhomogeneous dielectric medium, due to a point charge  $e$  at  $\mathbf{r}_A$ , is equal to the potential  $\phi_B(\mathbf{r}_A)$  at  $\mathbf{r}_A$  due to the same charge at  $\mathbf{r}_B$ .

**SOLUTION.** The potentials  $\phi_A(\mathbf{r})$  and  $\phi_B(\mathbf{r})$  satisfy the equations

$$\operatorname{div}(\epsilon \operatorname{grad} \phi_A) = -4\pi e \delta(\mathbf{r} - \mathbf{r}_A), \quad \operatorname{div}(\epsilon \operatorname{grad} \phi_B) = -4\pi e \delta(\mathbf{r} - \mathbf{r}_B).$$

Multiplying the first by  $\phi_B$  and the second by  $\phi_A$  and subtracting, we have

$$\operatorname{div}(\phi_B \epsilon \operatorname{grad} \phi_A) - \operatorname{div}(\phi_A \epsilon \operatorname{grad} \phi_B) = -4\pi e \delta(\mathbf{r} - \mathbf{r}_A) \phi_B(\mathbf{r}) + 4\pi e \delta(\mathbf{r} - \mathbf{r}_B) \phi_A(\mathbf{r}).$$

Integration of this equation over all space gives the required relation:

$$\phi_A(\mathbf{r}_B) = \phi_B(\mathbf{r}_A).$$

## §8. A dielectric ellipsoid

The polarization of a dielectric ellipsoid in a uniform external electric field has some unusual properties which render this example particularly interesting.

Let us consider first a simple special case, that of a dielectric sphere in an external field  $\mathcal{E}$ . We denote its permittivity by  $\epsilon^{(i)}$ , and that of the medium surrounding it by  $\epsilon^{(e)}$ . We take the origin of spherical polar coordinates at the centre of the sphere, and the direction of  $\mathcal{E}$  as the axis from which the polar angle  $\theta$  is measured, and seek the field potential outside the

sphere in the form  $\phi^{(e)} = \mathbf{E} \cdot \mathbf{r} + A\mathbf{E} \cdot \mathbf{r}/r^3$ ; the first term is the potential of the external field imposed, and the second, which vanishes at infinity, gives the required change in potential due to the sphere (cf. §3, Problem 1, solution). Inside the sphere, we seek the field potential in the form  $\phi^{(i)} = -B\mathbf{E} \cdot \mathbf{r}$ , the only function which satisfies Laplace's equation, remains finite at the centre of the sphere, and depends only on the constant vector  $\mathbf{E}$  (which is the only parameter of the problem).

The constants  $A$  and  $B$  are determined by the boundary conditions on the surface of the sphere. It may be seen at once, however, that the field in the sphere  $\mathbf{E}^{(i)} = B\mathbf{E}$  is uniform and differs only in magnitude from the applied field  $\mathbf{E}$ .

The boundary condition of continuity of the potential gives  $\mathbf{E}^{(i)} = \mathbf{E}(1 - A/R^3)$ , where  $R$  is the radius of the sphere, and the condition of continuity of the normal component of the induction gives

$$\mathbf{D}^{(i)} = \epsilon^{(e)}\mathbf{E}(1 + 2A/R^3).$$

Eliminating  $A$  from these two equations, we obtain

$$\frac{1}{3}(\mathbf{D}^{(i)} + 2\epsilon^{(e)}\mathbf{E}^{(i)}) = \epsilon^{(e)}\mathbf{E} \quad (8.1)$$

or, substituting  $\mathbf{D}^{(i)} = \epsilon^{(i)}\mathbf{E}^{(i)}$ ,

$$\mathbf{E}^{(i)} = 3\epsilon^{(e)}\mathbf{E}/(2\epsilon^{(e)} + \epsilon^{(i)}). \quad (8.2)$$

The problem of an infinite dielectric cylinder in an external field perpendicular to its axis is solved in an entirely similar manner (cf. §3, Problem 2). The field inside the cylinder, like that inside the sphere in the above example, is uniform. It satisfies the relation

$$\frac{1}{2}(\mathbf{D}^{(i)} + \epsilon^{(e)}\mathbf{E}^{(i)}) = \epsilon^{(e)}\mathbf{E}, \quad (8.3)$$

or

$$\mathbf{E}^{(i)} = 2\epsilon^{(e)}\mathbf{E}/(\epsilon^{(e)} + \epsilon^{(i)}). \quad (8.4)$$

The relations (8.1) and (8.3), in which the permittivity  $\epsilon^{(i)}$  of the sphere or cylinder does not appear explicitly, are particularly important because their validity does not depend on a linear relation between  $\mathbf{E}$  and  $\mathbf{D}$  within the body; they hold whatever the form of this relation (e.g. for anisotropic bodies). The analogous relations

$$\mathbf{E}^{(i)} = \mathbf{E} \quad (8.5)$$

for a cylinder in a longitudinal field and

$$\mathbf{D}^{(i)} = \epsilon^{(e)}\mathbf{E} \quad (8.6)$$

for a flat plate in a field perpendicular to it are similarly valid; these relations are evident at once from the boundary conditions.

The property of causing a uniform field within itself on being placed in a uniform external field is found to pertain to any ellipsoid, whatever the ratio of the semiaxes  $a, b, c$ . The problem of the polarization of a dielectric ellipsoid is solved by the use of ellipsoidal coordinates, in the same way as the corresponding problem for a conducting ellipsoid in §4.

Let the external field be again in the  $x$ -direction. The field potential outside the ellipsoid may again be sought in the form (4.22):  $\phi'_e = \phi_0 F(\xi)$ , with the function  $F(\xi)$  given by (4.23). Such a function cannot, however, appear in the field potential  $\phi_i$  inside the ellipsoid, since it does not satisfy the condition that the field must be finite everywhere inside the ellipsoid.

For let us consider the surface  $\xi = -c^2$ , which is an ellipse in the  $xy$ -plane, with semiaxes  $\sqrt{(a^2 - c^2)}$  and  $\sqrt{(b^2 - c^2)}$ , lying within the ellipsoid. For  $\xi \rightarrow -c^2$ , the integral (4.23) behaves as  $\sqrt{(\xi + c^2)}$ . The field, i.e. the potential gradient, therefore behaves as  $1/\sqrt{(\xi + c^2)}$ , and becomes infinite at  $\xi = -c^2$ . Thus the only solution suitable for the field inside the ellipsoid is  $F(\xi) = \text{constant}$ , so that  $\phi_i$  must be sought in the form  $\phi_i = B\phi_0$ . We see that the potential  $\phi_i$  differs only by a constant factor from the potential  $\phi_0$  of the uniform field. In other words, the field inside the ellipsoid is also uniform.

We shall not pause to write out the formulae for the field outside the ellipsoid. The uniform field inside the ellipsoid can be found without actually writing out the boundary conditions, by using some results already known.

Let us first suppose that the ellipsoid is in a vacuum ( $\epsilon^{(e)} = 1$ ). Then there must be a linear relation between the vectors  $\mathbf{E}^{(i)}$ ,  $\mathbf{D}^{(i)}$  and  $\mathbf{E}$  (which are all in the  $x$ -direction), of the form  $aE_x^{(i)} + bD_x^{(i)} = E_x$ , where the coefficients  $a, b$  depend only on the shape of the ellipsoid, and not on its permittivity  $\epsilon^{(i)}$ . The existence of such a relation follows from the form of the boundary conditions, as we saw above in the examples of the sphere and the cylinder.

To determine  $a$  and  $b$  we notice that, in the trivial particular case  $\epsilon^{(i)} = 1$ , we have simply  $\mathbf{E} = \mathbf{D} = \mathbf{E}$ , and so  $a + b = 1$ . Another particular case for which the solution is known is that of a conducting ellipsoid. In a conductor  $\mathbf{E}^{(i)} = 0$ , and the induction  $\mathbf{D}^{(i)}$ , though it has no direct physical significance, may be regarded formally as being related to the total dipole moment of the ellipsoid by  $\mathbf{D}^{(i)} = 4\pi\mathbf{P} = 4\pi\mathcal{P}/V$ . According to (4.26) we then have  $D_x^{(i)} = E_x/n^{(x)}$ , i.e.  $b = n^{(x)}$ , and so  $a = 1 - n^{(x)}$ . Thus we conclude that

$$(1 - n^{(x)})E_x^{(i)} + n^{(x)}D_x^{(i)} = E_x, \quad (8.7)$$

or

$$E_x^{(i)} = E_x - 4\pi n^{(x)}P_x. \quad (8.8)$$

The quantity  $4\pi n^{(x)}P_x$  is called the *depolarizing field*.† Similar relations, but with coefficients  $n^{(y)}, n^{(z)}$ , hold for the fields in the  $y$  and  $z$  directions. Like the particular formulae (8.1) and (8.3), they are valid whatever the relation between  $\mathbf{E}$  and  $\mathbf{D}$  inside the ellipsoid.

The field inside the ellipsoid is found from (8.7) by putting  $D_x^{(i)} = \epsilon^{(i)}E_x^{(i)}$ :

$$E_x^{(i)} = E_x / [1 + (\epsilon^{(i)} - 1)n^{(x)}], \quad (8.9)$$

and the total dipole moment of the ellipsoid is

$$\mathcal{P}_x = VP_x = (\epsilon^{(i)} - 1)VE_x^{(i)}/4\pi = \frac{1}{3}abc(\epsilon^{(i)} - 1)E_x / [1 + (\epsilon^{(i)} - 1)n^{(x)}]. \quad (8.10)$$

If the field  $\mathbf{E}$  has components along all three axes, then the field inside the ellipsoid is still uniform, but in general not parallel to  $\mathbf{E}$ . For an arbitrary choice of coordinate axes we can write the relation (8.7) in the general form

$$E_i^{(i)} + n_{ik}(D_k^{(i)} - E_k^{(i)}) = E_i. \quad (8.11)$$

The transition to the case where the permittivity of the medium differs from unity is effected by simply replacing  $\epsilon^{(i)}$  by  $\epsilon^{(i)}/\epsilon^{(e)}$ . Then formula (8.7) becomes

$$(1 - n^{(x)})\epsilon^{(e)}E_x^{(i)} + n^{(x)}D_x^{(i)} = \epsilon^{(e)}E_x. \quad (8.12)$$

† Similar formulae hold for a magnetized ellipsoid in a uniform external magnetic field (see §29). In this case  $n^{(x)}, n^{(y)}, n^{(z)}$  are called *demagnetizing factors*.

This formula can be applied, in particular, to the field inside an ellipsoidal cavity in an infinite dielectric medium. In this case  $\epsilon^{(i)} = 1$ .

### PROBLEMS†

PROBLEM 1. Determine the torque on a spheroid in a uniform electric field.

SOLUTION. According to the general formula (16.13), the torque on an ellipsoid is  $\mathbf{K} = \mathscr{P} \times \mathbf{E}$ , where  $\mathscr{P}$  is the dipole moment of the ellipsoid. In a spheroid, the vector  $\mathscr{P}$  is in a plane passing through the axis of symmetry and the direction of  $\mathbf{E}$ . The torque is perpendicular to this plane, and a calculation of its magnitude from formulae (8.10) gives

$$K = \frac{(\epsilon - 1)^2 |1 - 3n| E^2 V \sin 2\alpha}{8\pi [n\epsilon + 1 - n] [(1 - n)\epsilon + 1 + n]},$$

where  $\alpha$  is the angle between the direction of  $\mathbf{E}$  and the axis of symmetry of the spheroid, and  $n$  is the depolarizing factor along the axis (so that the depolarizing factors in the directions perpendicular to the axis are  $\frac{1}{2}(1 - n)$ ). The torque is directed so that it tends to turn the axis of symmetry of a prolate ( $n < \frac{1}{2}$ ) or oblate ( $n > \frac{1}{2}$ ) ellipsoid parallel or perpendicular to the field respectively.

For a conducting ellipsoid ( $\epsilon \rightarrow \infty$ ) we have

$$K = \frac{|1 - 3n|}{8\pi n(1 - n)} V E^2 \sin 2\alpha.$$

PROBLEM 2. A hollow dielectric sphere (with permittivity  $\epsilon$  and internal and external radii  $b$  and  $a$ ) is in a uniform external field  $\mathbf{E}$ . Determine the field in the cavity.

SOLUTION. As above in the problem of a solid sphere, we seek the field potentials in the vacuum outside the sphere (region 1) and in the cavity (region 3) in the forms  $\phi_1 = -E \cos \theta (r - A/r^2)$ ,  $\phi_3 = -B E r \cos \theta$ , and that in the dielectric (region 2) as  $\phi_2 = -C E \cos \theta (r - D/r^2)$ , where  $A, B, C, D$  are constants determined from the conditions of continuity of  $\phi$  and  $\epsilon \partial \phi / \partial r$  at the boundaries 1-2 and 2-3. Thus the field  $\mathbf{E}_3 = B E$  in the cavity is uniform, but the field  $\mathbf{E}_2$  in the sphere is not. A calculation of the constants gives the result

$$\mathbf{E}_3 = 9\epsilon E / [(\epsilon + 2)(2\epsilon + 1) - 2(\epsilon - 1)^2 (b/a)^3].$$

PROBLEM 3. The same as Problem 2, but for a hollow cylinder in a uniform transverse field.‡

SOLUTION. As in Problem 2, with the result

$$\mathbf{E}_3 = 4\epsilon E / [(\epsilon + 1)^2 - (\epsilon - 1)^2 (b/a)^2].$$

### §9. The permittivity of a mixture

If a substance is a finely dispersed mixture (an emulsion, powder mixture, etc.), we can consider the electric field averaged over volumes which are large compared with the scale of the inhomogeneities. The mixture is a homogeneous and isotropic medium with respect to such an average field, and so may be characterized by an effective permittivity, which we denote by  $\epsilon_{\text{mix}}$ . If  $\bar{\mathbf{E}}$  and  $\bar{\mathbf{D}}$  are the field and induction averaged in this way, then, by the definition of  $\epsilon_{\text{mix}}$ ,

$$\bar{\mathbf{D}} = \epsilon_{\text{mix}} \bar{\mathbf{E}}. \quad (9.1)$$

If all the particles in the mixture are isotropic, and the differences in their permittivities are small in comparison with  $\epsilon$  itself, it is possible to calculate  $\epsilon_{\text{mix}}$  in a general form which is correct as far as terms of the second order in these differences.

† In these three Problems the body is assumed to be in a vacuum.

‡ In a longitudinal field the solution is clearly  $\mathbf{E}_3 = \mathbf{E}$ .

We write the local field as  $\mathbf{E} = \bar{\mathbf{E}} + \delta\mathbf{E}$ , and the local permittivity as  $\bar{\epsilon} + \delta\epsilon$ , where

$$\bar{\epsilon} = (1/V) \int \epsilon dV \quad (9.2)$$

is obtained by averaging over the volume. Then the mean induction is

$$\bar{\mathbf{D}} = \overline{(\bar{\epsilon} + \delta\epsilon)(\bar{\mathbf{E}} + \delta\mathbf{E})} = \bar{\epsilon}\bar{\mathbf{E}} + \overline{\delta\epsilon\delta\mathbf{E}}, \quad (9.3)$$

since the mean values of  $\delta\epsilon$  and  $\delta\mathbf{E}$  are zero by definition. In the zero-order approximation  $\epsilon_{\text{mix}} = \bar{\epsilon}$ ; the first non-zero correction term will, of course, be of the second order in  $\delta\epsilon$ , as we see from (9.3).

From the non-averaged equation  $\text{div } \mathbf{D} = 0$  we have, as far as small terms of the first order,

$$\text{div}[(\bar{\epsilon} + \delta\epsilon)(\bar{\mathbf{E}} + \delta\mathbf{E})] = \bar{\epsilon}\text{div}\delta\mathbf{E} + \bar{\mathbf{E}} \cdot \text{grad } \delta\epsilon = 0. \quad (9.4)$$

The averaging of the product  $\delta\epsilon\delta\mathbf{E}$  in (9.3) is done in two stages. We first average over the volume of particles of a given kind, i.e. for a given  $\delta\epsilon$ . The value of  $\delta\mathbf{E}$  thus averaged is easily obtained from equation (9.4): on account of the isotropy of the mixture as a whole.

$$\frac{\partial}{\partial x} \overline{\delta E_x} = \frac{\partial}{\partial y} \overline{\delta E_y} = \frac{\partial}{\partial z} \overline{\delta E_z} = \frac{1}{3} \text{div } \delta\mathbf{E}.$$

If  $\bar{\mathbf{E}}$  is in the  $x$ -direction, say, we have from (9.4)

$$3\bar{\epsilon} \frac{\partial}{\partial x} \overline{\delta E_x} = -\bar{E}_x \frac{\partial \delta\epsilon}{\partial x},$$

whence

$$\overline{\delta E_x} = -(\bar{E}_x/3\bar{\epsilon})\delta\epsilon.$$

Since the direction of the  $x$ -axis is chosen arbitrarily, this equation may be written in the vector form  $\overline{\delta\mathbf{E}} = -(1/3\bar{\epsilon})\bar{\mathbf{E}}\delta\epsilon$ . Multiplying by  $\delta\epsilon$  and effecting the final averaging over all components of the mixture, we obtain  $\overline{\delta\epsilon\delta\mathbf{E}} = -(1/3\bar{\epsilon})\bar{\mathbf{E}}(\overline{\delta\epsilon})^2$ . Finally, substituting this expression in (9.3) and comparing with (9.1), we have the required result:

$$\epsilon_{\text{mix}} = \bar{\epsilon} - (1/3\bar{\epsilon})(\overline{\delta\epsilon})^2. \quad (9.5)$$

This formula can be written in another manner if we put

$$\bar{\epsilon}^{\frac{1}{3}} = \overline{(\bar{\epsilon} + \delta\epsilon)^{\frac{1}{3}}} = \bar{\epsilon}^{\frac{1}{3}} \left( 1 - \frac{(\overline{\delta\epsilon})^2}{9\bar{\epsilon}^2} \right);$$

this is accurate to terms of the second order. Then

$$\epsilon_{\text{mix}}^{\frac{1}{3}} = \bar{\epsilon}^{\frac{1}{3}}. \quad (9.6)$$

Thus we can say that, in this approximation, the cube root of  $\epsilon$  is additive.

Another limiting case for which an exact treatment is possible concerns the permittivity of an emulsion having an arbitrary difference between the permittivities  $\epsilon_1$  of the medium and  $\epsilon_2$  of the disperse phase but only a small concentration of the latter, whose particles are assumed spherical.

In the integral

$$\frac{1}{V} \int (\mathbf{D} - \epsilon_1 \mathbf{E}) dV \equiv \bar{\mathbf{D}} - \epsilon_1 \bar{\mathbf{E}}$$

the integrand is zero except within particles of the emulsion. It is therefore proportional to the volume concentration  $c$  of the emulsion, and in calculating it we can assume that the particles are in an external field which equals the mean field  $\bar{E}$ . Assuming the particles spherical and using formula (8.2), we obtain for the proportionality coefficient between  $\bar{D}$  and  $\bar{E}$

$$\epsilon_{\text{mix}} = \epsilon_1 + 3c\epsilon_1(\epsilon_2 - \epsilon_1)/(\epsilon_2 + 2\epsilon_1). \quad (9.7)$$

This formula is correct to terms of the first order in  $c$ . When  $\epsilon_1$  and  $\epsilon_2$  are nearly equal it is the same (to the first order in  $c$  and the second in  $\epsilon_2 - \epsilon_1$ ) as the result given by formula (9.5) for small  $c$ .

### §10. Thermodynamic relations for dielectrics in an electric field

The question of the change in thermodynamic properties owing to the presence of an electric field does not arise for conductors. Since there is no electric field inside a conductor, any change in its thermodynamic properties amounts simply to an increase in its total energy by the energy of the field which it produces in the surrounding space.† This quantity is quite independent of the thermodynamic state (and, in particular, of the temperature) of the body, and so does not affect the entropy, for example.

On the other hand, an electric field penetrates into a dielectric and so has a great effect on its thermodynamic properties. To investigate this effect, let us first determine the work done on a thermally insulated dielectric when the field in it undergoes an infinitesimal change.

The electric field in which the dielectric is placed must be imagined as due to various external charged conductors, and the change in the field can then be regarded as resulting from changes in the charges on these conductors.‡ Let us suppose for simplicity that there is only one conductor, with charge  $e$  and potential  $\phi$ . The work which must be done to increase its charge by an infinitesimal amount  $\delta e$  is

$$\delta R = \phi \delta e; \quad (10.1)$$

this is the mechanical work done by the given field on a charge  $\delta e$  brought from infinity (where the field potential is zero) to the surface of the conductor, i.e. through a potential difference of  $\phi$ . We shall put  $\delta R$  in a form which is expressed in terms of the field in the space filled with dielectric which surrounds the conductor.

If  $D_n$  is the component of the electric induction vector in the direction of the normal to the surface of the conductor (out of the dielectric and into the conductor), then the surface charge density on the conductor is  $-D_n/4\pi$ , so that

$$e = -\frac{1}{4\pi} \oint D_n df = -\frac{1}{4\pi} \oint \mathbf{D} \cdot d\mathbf{f}.$$

† We here neglect the energy of the attachment of the charge to the substance of the conductor; this will be discussed in §23.

‡ The final results which we shall obtain involve only the values of the field inside the dielectric, and therefore are independent of the origin of the field. For this reason there is no need for special discussion of the case where the field is produced, not by charged conductors, but (for instance) by extraneous charges placed in the dielectric itself or by pyroelectric polarization of it (§13).

Since the potential  $\phi$  is constant on the surface of the conductor, we can write

$$\delta R = \phi \delta e = -\frac{1}{4\pi} \oint \phi \delta \mathbf{D} \cdot d\mathbf{f} = -\frac{1}{4\pi} \int \text{div}(\phi \delta \mathbf{D}) dV.$$

The last integral is taken over the whole volume outside the conductor. Since the varied field, like the original field, must satisfy the field equations, we have  $\text{div} \delta \mathbf{D} = 0$ , and so  $\text{div}(\phi \delta \mathbf{D}) = \phi \text{div} \delta \mathbf{D} + \delta \mathbf{D} \cdot \text{grad} \phi = -\mathbf{E} \cdot \delta \mathbf{D}$ . Thus the following important formula is obtained:

$$\delta R = \int (\mathbf{E} \cdot \delta \mathbf{D} / 4\pi) dV. \quad (10.2)$$

It should be emphasized that the integration in (10.2) is over the whole field, including the vacuum if the dielectric does not occupy all space outside the conductor.

The work done on a thermally insulated body is just the change in its energy at constant entropy. Hence the expression (10.2) must be included in the thermodynamic relation which gives the infinitesimal change in the total energy of the body; the latter contains also the energy of the electric field. Denoting the total energy by  $\mathcal{U}$ , we therefore have

$$\delta \mathcal{U} = T \delta \mathcal{S} + \frac{1}{4\pi} \int \mathbf{E} \cdot \delta \mathbf{D} dV, \quad (10.3)$$

where  $T$  is the temperature of the body and  $\mathcal{S}$  its entropy.†

Accordingly we have for the total free energy‡  $\mathcal{F} = \mathcal{U} - T\mathcal{S}$

$$\delta \mathcal{F} = -\mathcal{S} \delta T + \frac{1}{4\pi} \int \mathbf{E} \cdot \delta \mathbf{D} dV. \quad (10.4)$$

Similar thermodynamic relations can be obtained for the quantities pertaining to unit volume of the body. Let  $U$ ,  $S$  and  $\rho$  be the internal energy, entropy and mass of unit volume. It is well known that the ordinary thermodynamic relation (in the absence of a field) for the internal energy of unit volume is  $dU = T dS + \zeta d\rho$ , where  $\zeta$  is the chemical potential of the substance.§ In the presence of a field in a dielectric, there must be added the integrand in (10.3):

$$dU = T dS + \zeta d\rho + \mathbf{E} \cdot d\mathbf{D} / 4\pi. \quad (10.5)$$

For the free energy per unit volume of the dielectric,  $F = U - TS$ , we therefore have

$$dF = -S dT + \zeta d\rho + \mathbf{E} \cdot d\mathbf{D} / 4\pi. \quad (10.6)$$

These relations are the basis of the thermodynamics of dielectrics.

We see that  $U$  and  $F$  are the thermodynamic potentials with respect to  $S$ ,  $\rho$ ,  $\mathbf{D}$  and  $T$ ,  $\rho$ ,  $\mathbf{D}$  respectively. In particular, we can obtain the field by differentiating these potentials with respect to the components of the vector  $\mathbf{D}$ :

$$\mathbf{E} = 4\pi(\partial U / \partial \mathbf{D})_{S, \rho} = 4\pi(\partial F / \partial \mathbf{D})_{T, \rho}. \quad (10.7)$$

† In (10.3) and (10.4) the volume of the body is assumed constant, but in general it becomes inhomogeneous in an electric field, and so the volume no longer characterizes the state of the body.

‡ This quantity is meaningful only when the temperature is constant throughout the body.

§ See SP 1, §24. Instead of the mass density we there use the number of particles  $N$  per unit volume;  $\rho = Nm$ , where  $m$  is the mass of one particle. The chemical potentials  $\zeta$  per unit mass and  $\mu$  per particle are related by  $\zeta = \mu/m$ .

The use of the letter  $\rho$  for the mass density as well as the charge density cannot lead to any misunderstanding, because the two quantities never appear together.

The free energy is more convenient in this respect, since it is to be differentiated at constant temperature, whereas the internal energy must be expressed in terms of the entropy, which is less easy.

Together with  $U$  and  $F$ , it is convenient to introduce thermodynamic potentials in which the components of the vector  $\mathbf{E}$ , instead of  $\mathbf{D}$ , are the independent variables. Such are

$$\tilde{U} = U - \mathbf{E} \cdot \mathbf{D}/4\pi, \quad \tilde{F} = F - \mathbf{E} \cdot \mathbf{D}/4\pi. \quad (10.8)$$

On differentiating these we have

$$\left. \begin{aligned} d\tilde{U} &= TdS + \zeta d\rho - \mathbf{D} \cdot d\mathbf{E}/4\pi, \\ d\tilde{F} &= -SdT + \zeta d\rho - \mathbf{D} \cdot d\mathbf{E}/4\pi. \end{aligned} \right\} \quad (10.9)$$

Hence, in particular,

$$\mathbf{D} = -4\pi(\partial\tilde{U}/\partial\mathbf{E})_{S,\rho} = -4\pi(\partial\tilde{F}/\partial\mathbf{E})_{T,\rho}. \quad (10.10)$$

It should be noticed that the relation between the thermodynamic quantities with and without the tilde is exactly that which occurs in §5 for the energy of the electrostatic field of conductors in a vacuum. For the integral  $\int \mathbf{E} \cdot \mathbf{D} dV$  can be transformed in an exactly similar manner to the one at the beginning of §2, with the equation  $\text{div } \mathbf{D} = 0$  inside the dielectric and the boundary condition  $D_n = 4\pi\sigma$  on the surfaces of conductors:

$$\begin{aligned} \frac{1}{4\pi} \int \mathbf{E} \cdot \mathbf{D} dV &= -\frac{1}{4\pi} \int \text{grad } \phi \cdot \mathbf{D} dV \\ &= \frac{1}{4\pi} \sum_a \int \phi_a D_n df = \sum_a \phi_a e_a. \end{aligned} \quad (10.11)$$

Hence we have for the internal energy, for example,

$$\tilde{\mathcal{U}} = \mathcal{U} - \int \frac{\mathbf{E} \cdot \mathbf{D}}{4\pi} dV = \mathcal{U} - \sum_a \phi_a e_a, \quad (10.12)$$

in agreement with the definition (5.5).

It is useful to derive also the formulae for infinitesimal changes in these quantities, expressed in terms of the charges and potentials of the conductors (the sources of the field). For example, the variation in the free energy (for a given temperature) is

$$(\delta \mathcal{F})_T = \delta R = \sum_a \phi_a \delta e_a. \quad (10.13)$$

For the variation of  $\tilde{\mathcal{F}}$  we have

$$(\delta \tilde{\mathcal{F}})_T = (\delta \mathcal{F})_T - \delta \sum_a \phi_a e_a = -\sum_a e_a \delta \phi_a. \quad (10.14)$$

We can say that the quantities without the tilde are the thermodynamic potentials with respect to the charges on the conductors, while those with it are thermodynamic potentials with respect to their potentials.

It is known from thermodynamics that the various thermodynamic potentials have the property of being minima in a state of thermodynamic equilibrium, relative to various changes in the state of the body. In formulating these conditions of equilibrium in an



electric field, it is necessary to say whether changes of state with constant charges on the conductors (the field sources) or those with constant potentials are being considered. For example, in equilibrium  $\mathcal{F}$  and  $\tilde{\mathcal{F}}$  are minima with respect to changes in state occurring at constant temperature and (respectively) constant charges and potentials of the conductors (the same is true for  $\mathcal{U}$  and  $\tilde{\mathcal{U}}$  at constant entropy).

If any processes (such as chemical reactions) which are not directly related to the electric field can occur in the body, the condition of equilibrium with respect to these processes is that  $F$  be a minimum for given density, temperature and induction  $\mathbf{D}$ , or that  $\tilde{F}$  be a minimum for constant density, temperature and field  $\mathbf{E}$ .

Hitherto we have made no assumptions concerning the dependence of  $\mathbf{D}$  on  $\mathbf{E}$ , so that all the thermodynamic relations derived above are valid whatever the nature of this dependence. Let us now apply them to an isotropic dielectric, where a linear relation  $\mathbf{D} = \epsilon \mathbf{E}$  holds. In this case integration of (10.5) and (10.6) gives

$$\left. \begin{aligned} U &= U_0(S, \rho) + D^2/8\pi\epsilon, \\ F &= F_0(T, \rho) + D^2/8\pi\epsilon. \end{aligned} \right\} \quad (10.15)$$

where  $U_0$  and  $F_0$  pertain to the dielectric in the absence of the field. Thus in this case the quantity

$$D^2/8\pi\epsilon = \epsilon E^2/8\pi = ED/8\pi \quad (10.16)$$

is the change in the internal energy (for given entropy and density) or in the free energy (for given temperature and density), per unit volume of the dielectric medium, resulting from the presence of the field.

The expressions for the potentials  $\tilde{U}$  and  $\tilde{F}$  are similarly

$$\left. \begin{aligned} \tilde{U} &= U_0(S, \rho) - \epsilon E^2/8\pi, \\ \tilde{F} &= F_0(T, \rho) - \epsilon E^2/8\pi. \end{aligned} \right\} \quad (10.17)$$

We see that the differences  $U - U_0$  and  $\tilde{U} - U_0$  in this case differ only in sign, as they did for an electric field in a vacuum (§5). In a dielectric medium, however, this simple result holds good only when there is a linear relation between  $\mathbf{D}$  and  $\mathbf{E}$ .

We shall write out also, for future reference, formulae for the entropy density  $S$  and the chemical potential  $\zeta$ , which follow from (10.15):

$$\begin{aligned} S &= -\left(\frac{\partial F}{\partial T}\right)_{\rho, \mathbf{D}} = S_0(T, \rho) + \frac{D^2}{8\pi\epsilon^2} \left(\frac{\partial \epsilon}{\partial T}\right)_{\rho} \\ &= S_0(T, \rho) + \frac{E^2}{8\pi} \left(\frac{\partial \epsilon}{\partial T}\right)_{\rho}, \end{aligned} \quad (10.18)$$

$$\zeta = \left(\frac{\partial F}{\partial \rho}\right)_{T, \mathbf{D}} = \zeta_0(T, \rho) - \frac{E^2}{8\pi} \left(\frac{\partial \epsilon}{\partial \rho}\right)_T. \quad (10.19)$$

These quantities, of course, differ from zero only inside the dielectric.

The total free energy is obtained by integrating (10.15) over all space. By (10.11) we have

$$\mathcal{F} - \mathcal{F}_0 = \int \mathbf{E} \cdot \mathbf{D} dV/8\pi = \frac{1}{2} \sum e_a \phi_a. \quad (10.20)$$

This last expression is formally identical with the energy of the electrostatic field of conductors in a vacuum. The same result can be obtained directly by starting from the variation  $\delta\mathcal{F}$  (10.13) for an infinitesimal change in the charges on the conductors. In the present case, when  $\mathbf{D}$  and  $\mathbf{E}$  are linearly related, all the field equations and their boundary conditions are also linear. Hence the potentials of the conductors must (as for the field in a vacuum) be linear functions of their charges, and integration of equation (10.13) gives (10.20).

It should be emphasized that these arguments do not presuppose the dielectric to fill all space outside the conductors. If, however, this is so, we can go further and use the results at the end of §7 to draw the following conclusion. For given charges on the conductors, the presence of the dielectric medium reduces by a factor  $\epsilon$  both the potentials of the conductors and the field energy, as compared with the values for a field in a vacuum. If, on the other hand, the potentials of the conductors are maintained constant, then their charges and the field energy are increased by a factor  $\epsilon$ .

### PROBLEM

Determine the height  $h$  to which a liquid rises in a vertical plane capacitor.

**SOLUTION.** For given potentials on the capacitor plates,  $\mathcal{F}$  must be a minimum.  $\mathcal{F}$  includes the energy  $\frac{1}{2}\rho gh^2$  of the liquid under gravity. From this condition we easily obtain  $h = (\epsilon - 1)E^2/8\pi\rho g$ .

### §11. The total free energy of a dielectric

The total free energy  $\mathcal{F}$  (or the total internal energy  $\mathcal{U}$ ), as defined in §10, includes the energy of the external electric field which polarizes the dielectric; this field may be imagined as being produced by a particular assembly of conductors with specified total charges. It is also meaningful to consider the total free energy less the energy of the field which would be present in all space if the body were absent. We denote this field by  $\mathcal{C}$ . Then the total free energy in this sense is

$$\int (F - \mathcal{C}^2/8\pi) dV, \quad (11.1)$$

where  $F$  is the free energy density. Here we shall denote this quantity by the letter  $\mathcal{F}$ , which in §10 signified  $\int F dV$ . It should be emphasized that the difference between the two definitions of  $\mathcal{F}$  is a quantity independent of the thermodynamic state and properties of the dielectric, and hence it has no effect on the fundamental differential relations of thermodynamics pertaining to this quantity.†

Let us calculate the change in  $\mathcal{F}$  resulting from an infinitesimal change in the field which occurs at constant temperature and does not destroy the thermodynamic equilibrium of the medium. Since  $\delta F = \mathbf{E} \cdot \delta \mathbf{D}/4\pi$ , we have  $\delta \mathcal{F} = \int (\mathbf{E} \cdot \delta \mathbf{D} - \mathcal{C} \cdot \delta \mathcal{C}) dV/4\pi$ . This expression is identically equal to

$$\delta \mathcal{F} = \int (\mathbf{D} - \mathcal{C}) \cdot \delta \mathcal{C} dV/4\pi + \int \mathbf{E} \cdot (\delta \mathbf{D} - \delta \mathcal{C}) dV/4\pi - \int (\mathbf{D} - \mathbf{E}) \cdot \delta \mathcal{C} dV/4\pi. \quad (11.2)$$

† It may be noted that there would be no sense in subtracting  $E^2/8\pi$  from  $F$ , because  $\mathbf{E}$  is the field as modified by the presence of the dielectric, and so the difference  $F - E^2/8\pi$  could not be regarded as the free energy density of the dielectric as such.

In the first integral we write  $\delta \mathbf{E} = -\text{grad } \delta \phi_0$  (where  $\phi_0$  is the potential of the field  $\mathbf{E}$ ) and integrate by parts:

$$\int \text{grad } \delta \phi_0 \cdot (\mathbf{D} - \mathbf{E}) dV = \oint \delta \phi_0 (\mathbf{D} - \mathbf{E}) \cdot d\mathbf{f} - \int \delta \phi_0 \text{div} (\mathbf{D} - \mathbf{E}) dV.$$

It is easy to see that both the integrals on the right-hand side are zero. For the volume integral this follows at once from the equations  $\text{div } \mathbf{D} = 0$  and  $\text{div } \mathbf{E} = 0$  which the induction in the dielectric and the field in the vacuum must respectively satisfy. The surface integral is taken over the surfaces of the conductors which produce the field and over an infinitely distant surface. The latter of these is, as usual, zero, and for each of the conductors  $\delta \phi_0 = \text{constant}$ , so that  $\oint \delta \phi_0 (\mathbf{D} - \mathbf{E}) \cdot d\mathbf{f} = \delta \phi_0 \oint (\mathbf{D} - \mathbf{E}) \cdot d\mathbf{f}$ . The field  $\mathbf{E}$ , by definition, is produced by the same sources as the field  $\mathbf{D}$  and induction  $\mathbf{D}$  (i.e. by the same conductors with given total charges  $e$ ). Hence the two integrals  $\oint \mathbf{D}_n d\mathbf{f}$  and  $\oint \mathbf{E}_n d\mathbf{f}$  are both equal to  $4\pi e$ , and their difference is zero.

Similarly, we can see that the second term in (11.2) is also zero, by putting  $\mathbf{E} = -\text{grad } \phi$  and using the same transformation. Finally, we have

$$\delta \mathcal{F} = - \int (\mathbf{D} - \mathbf{E}) \cdot \delta \mathbf{E} dV / 4\pi = - \int \mathbf{P} \cdot \delta \mathbf{E} dV. \quad (11.3)$$

It should be noticed that the integral in this expression need be taken only over the volume of the dielectric medium, since outside it  $\mathbf{P} = 0$ .

However, we must emphasize that the integrand  $\mathbf{P} \cdot \delta \mathbf{E}$  cannot be interpreted as the variation of the free energy density in the same way as was done with formulae (10.3), (10.4). First of all, this density must exist outside the body, which modifies the field in the surrounding space also. It is clear, moreover, that the energy density at any point in the body can depend only on the field actually present there, and not on the field which would be present if the body were removed.

If the external field  $\mathbf{E}$  is uniform, then

$$\delta \mathcal{F} = - \delta \mathbf{E} \cdot \int \mathbf{P} dV = - \mathcal{P} \cdot \delta \mathbf{E}, \quad (11.4)$$

where  $\mathcal{P}$  is the total electric dipole moment of the body. Hence the thermodynamic identity for the free energy can be written in this case as

$$d\mathcal{F} = - \mathcal{S} dT - \mathcal{P} \cdot d\mathbf{E}. \quad (11.5)$$

The total electric moment of the body can therefore be obtained by differentiating the total free energy:

$$\mathcal{P} = - (\partial \mathcal{F} / \partial \mathbf{E})_T. \quad (11.6)$$

The latter formula can also be obtained directly from the general statistical formula

$$\overline{\partial \hat{\mathcal{H}} / \partial \lambda} = (\partial \mathcal{F} / \partial \lambda)_T,$$

where  $\hat{\mathcal{H}}$  is the Hamiltonian of the body as the system of its component particles, and  $\lambda$  is any parameter characterizing the external conditions in which the body is placed; see SP 1, (11.4), (15.11). For a body in a uniform external field  $\mathbf{E}$ , the Hamiltonian contains a term  $-\mathbf{E} \cdot \hat{\mathcal{P}}$ , where  $\hat{\mathcal{P}}$  is the dipole moment operator. Taking  $\mathbf{E}$  as the parameter  $\lambda$ , we obtain the required formula.

If  $\mathbf{D}$  and  $\mathbf{E}$  are connected by the linear relation  $\mathbf{D} = \epsilon \mathbf{E}$ , we can similarly calculate explicitly not only the variation  $\delta \mathcal{F}$  but  $\mathcal{F}$  itself. We have

$$\mathcal{F} - \mathcal{F}_0 = \int (\mathbf{E} \cdot \mathbf{D} - \mathbf{E}^2) dV / 8\pi.$$

This can be identically transformed into

$$\mathcal{F} - \mathcal{F}_0 = \int (\mathbf{E} + \mathbf{E}) \cdot (\mathbf{D} - \mathbf{E}) dV / 8\pi - \int \mathbf{E} \cdot (\mathbf{D} - \mathbf{E}) dV / 8\pi.$$

The first term on the right is zero, as we see by putting

$$\mathbf{E} + \mathbf{E} = -\text{grad}(\phi + \phi_0)$$

and again using the same transformation. Hence we have

$$\mathcal{F} - \mathcal{F}_0(V, T) = -\frac{1}{2} \int \mathbf{E} \cdot \mathbf{P} dV. \quad (11.7)$$

In particular, in a uniform external field

$$\mathcal{F} - \mathcal{F}_0(V, T) = -\frac{1}{2} \mathbf{E} \cdot \mathcal{P}. \quad (11.8)$$

This last equation can also be obtained by direct integration of the relation (11.3) if we notice that, since all the field equations are linear when  $\mathbf{D} = \epsilon \mathbf{E}$ , the electric moment  $\mathcal{P}$  must be a linear function of  $\mathbf{E}$ .

The linear relation between the components of  $\mathcal{P}$  and  $\mathbf{E}$  can be written

$$\mathcal{P}_i = V \alpha_{ik} E_k, \quad (11.9)$$

as for conductors (§2). For a dielectric, however, the polarizability depends not only on the shape but also on the permittivity. The symmetry of the tensor  $\alpha_{ik}$ , mentioned in §2, follows at once from the relation (11.6); it is sufficient to notice that the second derivative  $\partial^2 \mathcal{F} / \partial E_k \partial E_i = -\partial \mathcal{P}_i / \partial E_k = -V \alpha_{ik}$  is independent of the order of differentiation.

Formula (11.7) becomes still simpler in the important case where  $\epsilon$  is close to 1, i.e. the dielectric susceptibility  $\kappa = (\epsilon - 1)/4\pi$  is small. In this case, in calculating the energy, we can neglect the modification of the field due to the presence of the body, putting  $\mathbf{P} = \kappa \mathbf{E} \cong \kappa \mathbf{E}$ . Then

$$\mathcal{F} - \mathcal{F}_0 = -\frac{1}{2} \kappa \int E^2 dV, \quad (11.10)$$

the integral being taken over the volume of the body. In a uniform field, the dipole moment  $\mathcal{P} = V \kappa \mathbf{E}$ , and the free energy is

$$\mathcal{F} - \mathcal{F}_0 = -\frac{1}{2} \kappa V E^2. \quad (11.11)$$

In the general case of an arbitrary relation between  $\mathbf{D}$  and  $\mathbf{E}$ , the simple formulae (11.7) and (11.8) do not hold. Here the formula

$$\mathcal{F} = \int \left( F - \frac{E^2}{8\pi} \right) dV = \int \left[ F - \frac{\mathbf{E} \cdot \mathbf{D}}{8\pi} - \frac{1}{2} \mathbf{P} \cdot \mathbf{E} \right] dV \quad (11.12)$$

may be useful in calculating  $\mathcal{F}$ ; its derivation is obvious after the above discussion. The two integrands differ by

$$-\frac{\mathbf{E} \cdot \mathbf{D}}{8\pi} - \frac{1}{2} \mathbf{E} \cdot \mathbf{P} + \frac{E^2}{8\pi} = -\frac{1}{8\pi} (\mathbf{D} - \mathbf{E}) \cdot (\mathbf{E} + \mathbf{E});$$

after substitution of  $\mathbf{E} = -\text{grad} \phi$ ,  $\mathbf{E} = -\text{grad} \phi_0$  and integration over all space, the result is zero. In (11.12), as in (11.7), the second integrand is zero outside the body, where  $\mathbf{P} = 0$  and  $F = E^2/8\pi$ , so that the integration is taken only over the volume of the body.

## PROBLEM

Derive the formula which replaces (11.7) when the body is not in a vacuum but in a medium of permittivity  $\epsilon^{(e)}$ .

SOLUTION. Using the same transformations as before, we find

$$\mathcal{F} - \mathcal{F}_0 = -\frac{1}{8\pi} \int \mathfrak{E} \cdot (\mathbf{D} - \epsilon^{(e)} \mathbf{E}) dV.$$

## §12. Electrostriction of isotropic dielectrics

For a solid dielectric in an electric field the concept of pressure cannot be defined as for an isotropic body in the absence of a field, because the forces acting on a dielectric (which we shall determine in §§15, 16) vary over the body, and are anisotropic even if the body itself is isotropic. An exact determination of the deformation (*electrostriction*) of such a body involves the solution of a complex problem in the theory of elasticity.

However, matters are much simpler if we are interested only in the change in the total volume of the body. As we saw in §5, the shape of the body may then be regarded as unchanged, i.e. the deformation may be regarded as a uniform volume compression or expansion.

We shall neglect the dielectric properties of the external medium (the atmosphere, for instance) in which the body is situated, i.e. we suppose that  $\epsilon = 1$ . This medium thus serves merely to exert a uniform pressure on the surface of the body, which we shall denote by  $P$ . If  $\mathcal{F}$  is the total free energy of the body, then we have the thermodynamic relation  $P = -(\partial \mathcal{F} / \partial V)_T$ , and accordingly the expression for the differential  $d\mathcal{F}$  contains a term  $-PdV$ . For example, in a uniform external field, (11.5) becomes

$$d\mathcal{F} = -\mathcal{S} dT - PdV - \mathcal{P} \cdot d\mathfrak{E}.$$

We introduce the total thermodynamic potential (Gibbs function) of the body in accordance with the usual thermodynamic relation

$$\mathcal{G} = \mathcal{F} + PV. \quad (12.1)$$

The differential of this quantity in a uniform external field is

$$d\mathcal{G} = -\mathcal{S} dT + V dP - \mathcal{P} \cdot d\mathfrak{E}. \quad (12.2)$$

The change in the thermodynamic quantities in an external electric field is usually a relatively small quantity. It is known from the theorem of small increments, SP 1 (15.12), that a small change in the free energy (for given  $T$  and  $V$ ) is equal to the small change in the thermodynamic potential (for given  $T$  and  $P$ ). Hence, besides (11.8), we can write analogously

$$\mathcal{G} = \mathcal{G}_0 - \frac{1}{2} \mathfrak{E} \cdot \mathcal{P} \quad (12.3)$$

for the thermodynamic potential of a body in a uniform external field. Here  $\mathcal{G}_0$  is the value for the body in the absence of the field and for given values of  $P$  and  $T$ , while  $\mathcal{F}_0$  in (11.8) is the free energy in the absence of the field and for given values of  $V$  and  $T$ .

Making explicit the dependence of the dipole moment on  $V$  and  $\mathfrak{E}$  according to (11.9), we can rewrite (12.3) as

$$\mathcal{G} = \mathcal{G}_0(P, T) - \frac{1}{2} V \alpha_{ik} \mathfrak{E}_i \mathfrak{E}_k, \quad (12.4)$$

where the correction term must be expressed as a function of temperature and pressure by

means of the equation of state for the body in the absence of the field. In particular, for a substance with small dielectric susceptibility this formula becomes simply

$$g\rho = g\rho_0(P, T) - \frac{1}{2}\kappa V \mathfrak{E}^2; \quad (12.5)$$

cf. (11.11).

The required change in volume  $V - V_0$  in the external field can now be obtained immediately by differentiating  $g\rho$  with respect to pressure for constant  $T$  and  $\mathfrak{E}$ . For example, from (12.5) we have

$$V - V_0 = -\frac{1}{2}\mathfrak{E}^2 [\partial(\kappa V)/\partial P]_T. \quad (12.6)$$

This quantity may be either positive or negative (whereas, in electrostriction of conductors, the volume is always greater in the presence of the field).

Similarly, we can calculate the amount of heat  $Q$  absorbed in a dielectric when an external electric field is isothermally applied (the external pressure being constant).† Differentiation of  $g\rho - g\rho_0$  with respect to temperature gives the change in the entropy of the body, and by multiplying this by  $T$  we obtain the required quantity of heat. For example, from (12.5) we obtain

$$Q = \frac{1}{2}\mathfrak{E}^2 T [\partial(\kappa V)/\partial T]_P. \quad (12.7)$$

Positive values of  $Q$  correspond to absorption of heat.

## PROBLEMS

**PROBLEM 1.** Determine the change in volume and the electrocaloric effect for a dielectric ellipsoid in a uniform electric field parallel to one of its axes.

**SOLUTION.** From formulae (12.3) and (8.10) we have

$$g\rho = g\rho_0 - \frac{V}{8\pi} \frac{\epsilon - 1}{n\epsilon + 1 - n} \mathfrak{E}^2.$$

The relative change in volume is found to be‡

$$\frac{V - V_0}{V} = \frac{\mathfrak{E}^2}{8\pi} \left[ \frac{\epsilon - 1}{n\epsilon + 1 - n} \frac{1}{K} - \frac{1}{(n\epsilon + 1 - n)^2} \left( \frac{\partial \epsilon}{\partial P} \right)_T \right],$$

and the electrocaloric effect

$$Q = \frac{TV\mathfrak{E}^2}{8\pi} \left[ \frac{\epsilon - 1}{n\epsilon + 1 - n} \alpha + \frac{1}{(n\epsilon + 1 - n)^2} \left( \frac{\partial \epsilon}{\partial T} \right)_P \right],$$

where  $1/K = -(1/V)(\partial V/\partial P)_T$  is the compressibility of the body, and  $\alpha = (1/V)(\partial V/\partial T)_P$  the thermal expansion coefficient.

In particular, for a plane disc in a field perpendicular to it,  $n = 1$ , so that

$$\begin{aligned} \frac{V - V_0}{V} &= \frac{\mathfrak{E}^2}{8\pi} \left[ \frac{\epsilon - 1}{\epsilon} \frac{1}{K} - \frac{1}{\epsilon^2} \left( \frac{\partial \epsilon}{\partial P} \right)_T \right], \\ Q &= \frac{TV\mathfrak{E}^2}{8\pi} \left[ \frac{\epsilon - 1}{\epsilon} \alpha + \frac{1}{\epsilon^2} \left( \frac{\partial \epsilon}{\partial T} \right)_P \right]. \end{aligned}$$

† If the body is thermally insulated, the application of the field results in a change of temperature  $\Delta T = -Q/\mathcal{C}_P$ , where  $\mathcal{C}_P$  is the heat capacity at constant pressure.

‡ In the limit  $\epsilon \rightarrow \infty$ , we find as the change in volume of a conducting ellipsoid  $(V - V_0)/V = \mathfrak{E}^2/8\pi K n$ . For a sphere,  $n = \frac{1}{3}$ , and we recover the result in §5, Problem 4.

For a similar disc (or any cylinder) in a longitudinal field,  $n = 0$ , and

$$\frac{V - V_0}{V} = \frac{\mathcal{E}^2}{8\pi} \left[ \frac{\epsilon - 1}{K} - \left( \frac{\partial \epsilon}{\partial P} \right)_T \right], \quad Q = \frac{TV\mathcal{E}^2}{8\pi} \left[ (\epsilon - 1)\alpha + \left( \frac{\partial \epsilon}{\partial T} \right)_P \right].$$

**PROBLEM 2.** Determine the difference between the heat capacity  $\mathcal{C}_\phi$  of a plane disc in a field perpendicular to it, with a constant potential difference between its faces, and the heat capacity  $\mathcal{C}_D$  at constant induction, the external pressure being maintained constant in each case.†

**SOLUTION.** According to the results of Problem 1, the entropy of the disc is

$$\mathcal{S} = - \left( \frac{\partial \mathcal{G}_0}{\partial T} \right)_{P, \mathcal{E}} = \mathcal{S}_0(P, T) + \frac{V\mathcal{E}^2}{8\pi} \left[ \frac{\epsilon - 1}{\epsilon} \alpha + \frac{1}{\epsilon^2} \left( \frac{\partial \epsilon}{\partial T} \right)_P \right].$$

The induction inside the disc is the same as the external field:  $D = \mathcal{E}$ . Hence, to calculate the heat capacity  $\mathcal{C}_D$ , we must differentiate  $\mathcal{S}$  for constant  $\mathcal{E}$ . The potential difference between the faces of the disc is  $\phi = El = \mathcal{E}l/\epsilon$ , where  $l$  is its thickness. For a uniform compression or expansion of a body,  $l$  is proportional to  $V^{1/3}$ . Hence, to calculate the heat capacity  $\mathcal{C}_\phi$ , we must differentiate  $\mathcal{S}$  for constant  $\mathcal{E}V^{1/3}/\epsilon$ . The required difference is found to be

$$\mathcal{C}_\phi - \mathcal{C}_D = \frac{TV\mathcal{E}^2}{4\pi\epsilon} \left[ (\epsilon - 1)\alpha + \frac{1}{\epsilon} \left( \frac{\partial \epsilon}{\partial T} \right)_P \right] \left[ \frac{1}{\epsilon} \left( \frac{\partial \epsilon}{\partial T} \right)_P - \frac{1}{3}\alpha \right].$$

**PROBLEM 3.** Determine the electrocaloric effect in a homogeneous dielectric whose total volume is kept constant.

**SOLUTION.** Strictly speaking, when an external field is applied the density of the body changes (and ceases to be uniform), even if the total volume is kept constant. In calculating the change in the total entropy, however, we can ignore this and assume the density  $\rho$  constant at every point.‡

According to (10.18) the total entropy of the body is

$$\mathcal{S} = \mathcal{S}_0(\rho, T) + \frac{1}{8\pi} \left( \frac{\partial \epsilon}{\partial T} \right)_\rho \int E^2 dV,$$

where the integration is over the volume of the body. The amount of heat absorbed is

$$Q = \frac{T}{8\pi} \left( \frac{\partial \epsilon}{\partial T} \right)_\rho \int E^2 dV.$$

**PROBLEM 4.** Determine the difference  $\mathcal{C}_\phi - \mathcal{C}_D$  (see Problem 2) when the total volume of the disc is kept constant.

**SOLUTION.** When the volume, and therefore the thickness, of the disc are constant, differentiation for constant potential difference is equivalent to differentiation for constant field  $E$ . Using the formula of Problem 3 for the entropy we have

$$\mathcal{C}_E - \mathcal{C}_D = \frac{TVE^2}{4\pi\epsilon} \left( \frac{\partial \epsilon}{\partial T} \right)_\rho^2 = \frac{TV\mathcal{E}^2}{4\pi\epsilon^3} \left( \frac{\partial \epsilon}{\partial T} \right)_\rho^2.$$

**PROBLEM 5.** A capacitor consists of two conducting surfaces at a distance  $h$  apart which is small compared with their dimensions; the space between them is filled with a substance of permittivity  $\epsilon_1$ . A sphere of radius  $a \ll h$  and permittivity  $\epsilon_2$  is placed in the capacitor. Determine the change in capacitance.

**SOLUTION.** Let the sphere be placed in the capacitor in such a way that the potential difference  $\phi$  between the plates remains unchanged. The free energy for constant potentials of the conductors is  $\mathcal{F}$ . In the absence of the sphere,  $\mathcal{F} = -\frac{1}{2}C_0\phi^2$ , where  $C_0$  is the original capacitance. Since the sphere is small, we may imagine it to be brought into a uniform field  $\mathcal{E} = \phi/h$ , and the change in  $\mathcal{F}$  is small. The small change in  $\mathcal{F}$  at constant potentials is equal to the small change in  $\mathcal{F}$  at constant charges on the sources of the field. Using the formula derived in §11, Problem, and (8.2), we have

$$\mathcal{F} = -\frac{1}{2}C_0\phi^2 - \frac{1}{2}a^3\epsilon_1(\epsilon_2 - \epsilon_1)\phi^2/(2\epsilon_1 + \epsilon_2)h^2.$$

†  $\mathcal{C}_\phi$  is the heat capacity of a disc between the plates of a plane capacitor in circuit with a constant e.m.f. In an unconnected capacitor with constant charges on the plates, the heat capacity of the disc is  $\mathcal{C}_D$ .

‡ The change in density  $\delta\rho$  is of the second order with respect to the field ( $\propto E^2$ ), and the consequent change in the total entropy is of the fourth order: the term in the change of total entropy which is linear in  $\delta\rho$  is  $(\partial S_0/\partial\rho)\int\delta\rho dV$ , and the integral is zero because the total mass of the body is unaltered.

whence the required capacitance is

$$C = C_0 + a^3 \epsilon_1 (\epsilon_2 - \epsilon_1) / (2\epsilon_1 + \epsilon_2) h^2.$$

### §13. Dielectric properties of crystals

In an anisotropic dielectric medium (a single crystal) the linear relation between the electric induction and the electric field is less simple, and does not reduce to a simple proportionality.

The most general form of such a relation is

$$D_i = D_{0i} + \epsilon_{ik} E_k, \quad (13.1)$$

where  $\mathbf{D}_0$  is a constant vector, and the quantities  $\epsilon_{ik}$  form a tensor of rank two, called the *permittivity tensor* (or the *dielectric tensor*). The inhomogeneous term  $\mathbf{D}_0$  in (13.1) does not, however, appear for all crystals. The majority of the types of crystal symmetry do not admit this constant vector (see below), and we then have simply

$$D_i = \epsilon_{ik} E_k. \quad (13.2)$$

The tensor  $\epsilon_{ik}$  is symmetrical:

$$\epsilon_{ik} = \epsilon_{ki}. \quad (13.3)$$

In order to prove this, it is sufficient to use the thermodynamic relation (10.10) and to observe that the second derivative  $-4\pi \partial^2 \tilde{F} / \partial E_k \partial E_i = \partial D_i / \partial E_k = \epsilon_{ik}$  is independent of the order of differentiation.

For  $\tilde{F}$  itself we have (when (13.2) holds) the expression

$$\tilde{F} = F_0 - \epsilon_{ik} E_i E_k / 8\pi. \quad (13.4)$$

The free energy  $F$  is

$$F = \tilde{F} + E_i D_i / 4\pi = F_0 + \epsilon^{-1}_{ik} D_i D_k / 8\pi. \quad (13.5)$$

Like every symmetrical tensor of rank two, the tensor  $\epsilon_{ik}$  can be brought to diagonal form by a suitable choice of the coordinate axes. In general, therefore, the tensor  $\epsilon_{ik}$  is determined by three independent quantities, namely the three principal values  $\epsilon^{(1)}, \epsilon^{(2)}, \epsilon^{(3)}$ . All these are necessarily greater than unity, just as  $\epsilon > 1$  for an isotropic body (see §14).

The number of different principal values of the tensor  $\epsilon_{ik}$  may be less than three for certain symmetries of the crystal.

In crystals of the triclinic, monoclinic and orthorhombic systems, all three principal values are different; such crystals are said to be *biaxial*.† In crystals of the triclinic system, the directions of the principal axes of the tensor  $\epsilon_{ik}$  are not uniquely related to any directions in the crystal. In those of the monoclinic system, one of the principal axes must coincide with the twofold axis of symmetry or be perpendicular to the plane of symmetry of the crystal. In crystals of the orthorhombic system, all three principal axes of the tensor  $\epsilon_{ik}$  are crystallographically fixed.

Next, in crystals of the tetragonal, rhombohedral and hexagonal systems, two of the

† This name refers to the optical properties of the crystals; see §§98, 99.



three principal values are equal, so that there are only two independent quantities; such crystals are said to be *uniaxial*. One of the principal axes coincides with the fourfold, threefold or sixfold axis of crystal symmetry, but the directions of the other two principal axes can be chosen arbitrarily.

Finally, in crystals of the cubic system all three principal values of the tensor  $\epsilon_{ik}$  are the same, and the directions of the principal axes are entirely arbitrary. This means that the tensor  $\epsilon_{ik}$  is of the form  $\epsilon\delta_{ik}$ , i.e. it is determined by a single scalar  $\epsilon$ . In other words, as regards their dielectric properties, crystals of the cubic system are no different from isotropic bodies.

All these fairly obvious symmetry properties of the tensor  $\epsilon_{ik}$  become particularly clear if we use a concept from tensor algebra, the *tensor ellipsoid*, the lengths of whose semiaxes are proportional to the principal values of a symmetrical tensor of rank two. The symmetry of the ellipsoid corresponds to that of the crystal. For instance, in a uniaxial crystal the tensor ellipsoid degenerates into a spheroid completely symmetrical about the longitudinal axis; it should be emphasized that, as regards the physical properties of the crystal which are determined by a symmetrical tensor of rank two, the presence of a threefold or higher axis of symmetry is equivalent to complete isotropy in the plane perpendicular to this axis. In cubic crystals, the tensor ellipsoid degenerates into a sphere.

Let us now examine the dielectric properties of crystals for which the constant term  $\mathbf{D}_0$  appears in (13.1). The presence of this term signifies that the dielectric is spontaneously polarized even in the absence of an external electric field. Such bodies are said to be *pyroelectric*. The magnitude of this spontaneous polarization is, however, in practice always very small (in comparison with the molecular fields). This is because large values of  $\mathbf{D}_0$  would lead to strong fields within the body, which is energetically very unfavourable and therefore could not correspond to thermodynamic equilibrium. The smallness of  $\mathbf{D}_0$  also ensures the legitimacy of an expansion of  $\mathbf{D}$  in powers of  $\mathbf{E}$ , of which (13.1) represents the first two terms.

The thermodynamic quantities for a pyroelectric body are found by integrating the relation  $-4\pi\partial\tilde{F}/\partial E_i = D_i = D_{0i} + \epsilon_{ik}E_k$ , whence

$$\tilde{F} = F_0 - \epsilon_{ik}E_iE_k/8\pi - E_iD_{0i}/4\pi. \quad (13.6)$$

The free energy is

$$\begin{aligned} F &= \tilde{F} + E_iD_i/4\pi = F_0 + \epsilon_{ik}E_iE_k/8\pi \\ &= F_0 + \epsilon^{-1}_{ik}(D_i - D_{0i})(D_k - D_{0k})/8\pi. \end{aligned} \quad (13.7)$$

It should be noted that the term in  $\tilde{F}$  linear in  $E_i$  does not appear in  $F$ .†

The total free energy of a pyroelectric can be calculated from formula (11.12) by substituting (13.7) and (13.1). If there is no external field,  $\mathcal{E} = 0$ , and we have simply

$$\mathcal{F} = \int [F_0 - (\mathbf{E} \cdot \mathbf{D}_0/8\pi)] dV. \quad (13.8)$$

It is remarkable that the free energy of a pyroelectric in the absence of an external field depends, like the field  $\mathbf{E}$ , not only on the volume of the body but also on its shape.

† It should also be noted that in these formulae we neglect the *piezoelectric effect*, i.e. the effect of internal stresses on the electric properties of a body; see §17. The formulae given here are therefore, strictly speaking, applicable only when the fields are uniform throughout the body, and internal stresses do not arise.

As has already been pointed out, the phenomenon of pyroelectricity is not possible for every crystal symmetry. Since, in any symmetry transformation, all the properties of the crystal must remain unchanged, it is clear that the only crystals which can be pyroelectric are those in which there is a direction which is unchanged (and, in particular, not reversed) in all symmetry transformations, and that this will be the direction of the constant vector  $\mathbf{D}_0$ .

This condition is satisfied only by those symmetry groups which consist of a single axis together with planes of symmetry which pass through the axis. In particular, crystals having a centre of symmetry certainly cannot be pyroelectric. We may enumerate those out of the 32 crystal classes in which pyroelectricity occurs:

triclinic system:  $C_1$

monoclinic system:  $C_s, C_2$

orthorhombic system:  $C_{2v}$

tetragonal system:  $C_4, C_{4v}$

rhombohedral system:  $C_3, C_{3v}$

hexagonal system:  $C_6, C_{6v}$ .

There are, of course, no pyroelectric cubic crystals. In a crystal of class  $C_1$  the direction of the pyroelectric vector  $\mathbf{D}_0$  is not related to any direction fixed in the crystal; in one of class  $C_s$ , it must lie in the plane of symmetry. In all the remaining classes listed above the direction of  $\mathbf{D}_0$  is that of the axis of symmetry.†

It should be mentioned that, under ordinary conditions, pyroelectric crystals have zero total electric dipole moment, although their polarization is not zero. The reason is that there is a non-zero field  $\mathbf{E}$  inside a spontaneously polarized dielectric. Since a body usually has a small but non-zero conductivity, the presence of a field gives rise to a current, which flows until the free charges formed on the surface of the body annihilate the field inside it. The same effect is produced by ions deposited on the surface from the air. Experimentally, pyroelectric properties are observed when a body is heated and a change in its spontaneous polarization is detected.

## PROBLEMS

PROBLEM 1. Determine the field of a pyroelectric sphere in a vacuum.

SOLUTION. The field inside the sphere is uniform, and the field and induction are related by  $2\mathbf{E} = -\mathbf{D}$  (as follows from (8.1) when  $\mathcal{E} = 0$ , i.e. when there is no applied external field). Substituting in (13.1), we obtain the equation  $2E_i + \epsilon_{ik}E_k = -D_{0i}$ . We take the coordinate axes to be the principal axes of the tensor  $\epsilon_{ik}$ . Then this equation gives  $E_i = -D_{0i}/(2 + \epsilon^{(i)})$ . The polarization of the sphere is  $P_i = (D_i - E_i)/4\pi = 3D_{0i}/4\pi(2 + \epsilon^{(i)})$ . The field outside the sphere is that of an electric dipole with moment  $\mathcal{P} = \mathbf{P}V$ .

PROBLEM 2. Determine the field of a point charge in a homogeneous anisotropic medium.‡

SOLUTION. The field of a point charge is given by the equation  $\text{div } \mathbf{D} = 4\pi e\delta(\mathbf{r})$  (the charge being at the origin). In an anisotropic medium  $D_i = \epsilon_{ik}E_k = -\epsilon_{ik}\partial\phi/\partial x_k$ ; taking the coordinate axes  $x, y, z$  along the principal axes of the tensor  $\epsilon_{ik}$ , we obtain for the potential the equation

$$\epsilon^{(x)}\partial^2\phi/\partial x^2 + \epsilon^{(y)}\partial^2\phi/\partial y^2 + \epsilon^{(z)}\partial^2\phi/\partial z^2 = -4\pi e\delta(\mathbf{r}).$$

† In referring to symmetry conditions, we are regarding the crystal as an infinite medium. For a finite crystal, the exact value of the total dipole moment may depend (in an ionic crystal) on which crystal planes form its faces and whether these planes contain ions of only one sign or are electrically neutral. However, in macroscopic electrodynamics, which implies averaging over physically infinitesimal volumes, it is reasonable to consider that the position of the faces relative to the crystal lattice is averaged also. In consequence of this averaging,  $\mathbf{D}_0$  vanishes in any non-pyroelectric finite crystal, and in a pyroelectric one is independent of the face configuration.

‡ In Problems 2-6 the anisotropic dielectric is assumed to be non-pyroelectric.

By the introduction of new variables

$$x' = x/\sqrt{\epsilon^{(x)}}, \quad y' = y/\sqrt{\epsilon^{(y)}}, \quad z' = z/\sqrt{\epsilon^{(z)}}, \quad (1)$$

this becomes

$$\Delta' \phi = -\frac{4\pi e}{\sqrt{(\epsilon^{(x)}\epsilon^{(y)}\epsilon^{(z)})}} \delta(\mathbf{r}')$$

which formally differs from the equation for the field in a vacuum only in that  $e$  is replaced by  $e' = e/\sqrt{(\epsilon^{(x)}\epsilon^{(y)}\epsilon^{(z)})}$ . Hence

$$\phi = \frac{e'}{r'} = \frac{e}{\sqrt{(\epsilon^{(x)}\epsilon^{(y)}\epsilon^{(z)})}} \left[ \frac{x^2}{\epsilon^{(x)}} + \frac{y^2}{\epsilon^{(y)}} + \frac{z^2}{\epsilon^{(z)}} \right]^{-\frac{1}{2}}.$$

In tensor notation, independent of the system of coordinates chosen, we have

$$\phi = e/\sqrt{(|\epsilon| \epsilon^{-1}_{ik} x_i x_k)},$$

where  $|\epsilon|$  is the determinant of the tensor  $\epsilon_{ik}$ .

**PROBLEM 3.** Determine the capacitance of a conducting sphere, with radius  $a$ , in an anisotropic dielectric medium.

**SOLUTION.** By the transformation shown in Problem 2, the determination of the field of a sphere with charge  $e$  in an anisotropic medium reduces to the determination of the field in a vacuum due to a charge  $e'$  distributed over the surface of the ellipsoid  $\epsilon_{ik} x'_i x'_k = \epsilon^{(x)} x'^2 + \epsilon^{(y)} y'^2 + \epsilon^{(z)} z'^2 = a^2$ . Using formula (4.14) for the potential due to an ellipsoid, we find the required capacitance to be given by

$$\frac{1}{C} = \frac{1}{2\sqrt{(\epsilon^{(x)}\epsilon^{(y)}\epsilon^{(z)})}} \int_0^\infty \left[ \left( \xi + \frac{a^2}{\epsilon^{(x)}} \right) \left( \xi + \frac{a^2}{\epsilon^{(y)}} \right) \left( \xi + \frac{a^2}{\epsilon^{(z)}} \right) \right]^{-\frac{1}{2}} d\xi.$$

**PROBLEM 4.** Determine the field in a flat anisotropic plate in a uniform external field  $\mathfrak{E}$ .

**SOLUTION.** From the condition of continuity of the tangential component of the field it follows that  $\mathbf{E} = \mathfrak{E} + A\mathbf{n}$ , where  $\mathbf{E}$  is the uniform field inside the plate,  $\mathbf{n}$  a unit vector normal to its surface, and  $A$  a constant. The constant is determined from the condition of continuity of the normal component of the induction,  $\mathbf{n} \cdot \mathbf{D} = \mathbf{n} \cdot \mathfrak{E}$ , or  $n_i \epsilon_{ik} E_k = n_i \epsilon_{ik} \mathfrak{E}_k + A \epsilon_{ik} n_i n_k = \mathfrak{E}_i n_i$ . Hence  $A = -(\epsilon_{ik} - \delta_{ik}) n_i \mathfrak{E}_k / \epsilon_{lm} n_l n_m$ .

In particular, if the external field is along the normal to the plate (the  $z$ -direction), we have

$$A = \mathfrak{E}(1 - \epsilon_{zz})/\epsilon_{zz}.$$

If it is parallel to the plate and in the  $x$ -direction, then

$$A = -\mathfrak{E} \epsilon_{zx} / \epsilon_{zz}.$$

**PROBLEM 5.** Determine the torque on an anisotropic dielectric sphere, with radius  $a$ , in a uniform external field  $\mathfrak{E}$  in a vacuum.

**SOLUTION.** According to (8.2) we have for the field inside the sphere  $E_x = 3\mathfrak{E}_x/(\epsilon^{(x)} + 2)$ , and similarly for  $E_y, E_z$ . Here the axes of  $x, y, z$  are taken to be the principal axes of the tensor  $\epsilon_{ik}$ . Hence the components of the dipole moment of the sphere are

$$\mathcal{P}_x = \frac{4}{3} \pi a^3 P_x = \frac{\epsilon^{(x)} - 1}{\epsilon^{(x)} + 2} a^3 \mathfrak{E}_x, \text{ etc.}$$

The components of the torque on the sphere are

$$K_z = (\mathcal{P} \times \mathfrak{E})_z = 3a^3 \mathfrak{E}_x \mathfrak{E}_y (\epsilon^{(x)} - \epsilon^{(y)}) / (\epsilon^{(x)} + 2)(\epsilon^{(y)} + 2),$$

and similarly for  $K_x, K_y$ .

**PROBLEM 6.** An infinite anisotropic medium contains a spherical cavity with radius  $a$ . Express the field in the cavity in terms of the uniform field  $E^{(e)}$  far from the cavity.

**SOLUTION.** The transformation (1) of Problem 2 reduces the equation for the field potential in the medium to Laplace's equation for the field in a vacuum. The equation for the field potential in the cavity is transformed into that for the potential in a medium with permittivities  $1/\epsilon^{(x)}, 1/\epsilon^{(y)}, 1/\epsilon^{(z)}$ . Moreover, the sphere is transformed into

an ellipsoid with semiaxes  $a/\sqrt{\epsilon^{(x)}}$ ,  $a/\sqrt{\epsilon^{(y)}}$ ,  $a/\sqrt{\epsilon^{(z)}}$ . Let  $n^{(x)}$ ,  $n^{(y)}$ ,  $n^{(z)}$  be the depolarizing factors of such an ellipsoid (given by formulae (4.25)). Applying formula (8.7) to the field of this ellipsoid, we obtain the relation

$$(1 - n^{(x)}) \frac{\partial \phi^{(i)}}{\partial x'} + \frac{n^{(x)}}{\epsilon^{(x)}} \frac{\partial \phi^{(i)}}{\partial x'} = \frac{\partial \phi^{(e)}}{\partial x'},$$

and similarly for the  $y'$  and  $z'$  directions. Returning to the original coordinates, we have  $\partial \phi / \partial x' = \sqrt{\epsilon^{(x)}} \partial \phi / \partial x = -\sqrt{\epsilon^{(x)}} E_x$ , so that the field in the cavity is

$$E_x^{(i)} = \frac{\epsilon^{(x)}}{\epsilon^{(x)} - n^{(x)}(\epsilon^{(x)} - 1)} E_x^{(e)}.$$

#### §14. The sign of the dielectric susceptibility

To elucidate the way in which the thermodynamic quantities for a dielectric in a field depend on its permittivity, let us consider the formal problem of the change in the electric component of the total free energy of the body when  $\epsilon$  undergoes an infinitesimal change.

For an isotropic (not necessarily homogeneous) body we have by (10.20)  $\mathcal{F} - \mathcal{F}_0 = \int (D^2/8\pi\epsilon) dV$ . When  $\epsilon$  changes, so does the induction, and the variation in the free energy is therefore

$$\delta \mathcal{F} = \int \frac{\mathbf{D} \cdot \delta \mathbf{D}}{8\pi\epsilon} dV - \int \frac{D^2}{8\pi\epsilon^2} \delta\epsilon dV.$$

The first term on the right is the same as (10.2), which gives the work done in an infinitesimal change in the field sources (i.e. charges on conductors). In the present case, however, we are considering a change in the field but no change in the sources. This term therefore vanishes, leaving

$$\delta \mathcal{F} = - \int (\delta\epsilon/\epsilon^2) (D^2/8\pi) dV = - \int \delta\epsilon (E^2/8\pi) dV. \quad (14.1)$$

From this formula it follows that any increase in the permittivity of the medium, even if in only a part of it (the sources of the field remaining unchanged), reduces the total free energy. In particular, we can say that the free energy is always reduced when uncharged conductors are brought into a dielectric medium, since these conductors may (in electrostatics) be regarded as bodies whose permittivity is infinite. This conclusion generalizes the theorem (§2) that the energy of the electrostatic field in a vacuum is diminished when an uncharged conductor is placed in it.

The total free energy is diminished also when any charge is brought up to a dielectric body from infinity (a process which may be regarded as an increase of  $\epsilon$  in a certain volume of the field round the charge). In order to conclude from this that any charge is attracted to a dielectric, we should, strictly speaking, prove also that  $\mathcal{F}$  cannot attain a minimum for any finite distance between the charge and the body. We shall not pause here to prove this statement, especially as the presence of an attractive force between a charge and a dielectric may be regarded as a fairly evident consequence of the interaction between the charge and the dipole moment of the dielectric, which it polarizes.

We can deduce immediately from formula (14.1) the direction of motion of a dielectric body in an almost uniform electric field, i.e. one which may be regarded as uniform over the dimensions of the body. In this case  $E^2$  is taken outside the integral, and the difference  $\mathcal{F} - \mathcal{F}_0$  is a negative quantity, proportional to  $E^2$ . In order to take a position in which its free energy is a minimum, the body will therefore move in the direction of  $E$  increasing.

It can be shown independently of (14.1) that the total change in the free energy of a

dielectric when it is placed in an electric field is negative.† This can be done by the use of thermodynamic perturbation theory, the change in the free energy of the body being regarded as the result of a perturbation of its quantum energy levels by the external electric field. According to this theory we have

$$\mathcal{F} - \mathcal{F}_0 = \bar{V}_{nn} - \frac{1}{2} \sum_n \sum_m' \frac{|V_{nm}|^2 (w_m - w_n)}{E_n^{(0)} - E_m^{(0)}} - \frac{1}{2T} (\bar{V}_{nn} - \bar{V}_{nn})^2; \quad (14.2)$$

see SP 1, (32.6). Here  $E_n^{(0)}$  are the unperturbed levels,  $V_{nm}$  the matrix elements of the perturbing energy, and the bar denotes a statistical averaging with respect to the Gibbs distribution  $w_n = \exp\{(\mathcal{F}_0 - E_n^{(0)})/T\}$ .

The term  $\bar{V}_{nn}$  in formula (14.2), which is linear in the field, is zero except in pyroelectric bodies. The quadratic change in the free energy, which is of interest here, is given by the remaining terms. It is evident that they are negative.

On the other hand, it is clear from the derivation of (14.2) that the total free energy  $\mathcal{F}$  must be taken in this formula as described in §11, omitting the energy of the field which would exist in the absence of the body. The difference  $\mathcal{F} - \mathcal{F}_0$  is therefore given by the thermodynamic formula (11.7). Let us consider a long narrow cylinder placed parallel to a uniform external field  $\mathcal{E}$ . The field within the cylinder is then  $\mathcal{E}$  also, and its polarization  $\mathbf{P} = (\epsilon - 1)\mathcal{E}/4\pi$ , so that

$$\mathcal{F} - \mathcal{F}_0 = -(\epsilon - 1)V\mathcal{E}^2/8\pi.$$

Thus  $\mathcal{F} - \mathcal{F}_0$  is negative only if  $\epsilon > 1$ . This leads to the conclusion mentioned in §7 and already made use of, namely that the permittivity of all bodies exceeds unity, and the dielectric susceptibility  $\kappa = (\epsilon - 1)/4\pi$  is therefore positive.

In the same way we can prove the inequalities  $\epsilon^{(i)} > 1$  for the principal values of the tensor  $\epsilon_{ik}$  in an anisotropic dielectric medium. To do so, it is sufficient to consider the energy of a field parallel to each of the three principal axes in turn.

## §15. Electric forces in a fluid dielectric

The problem of calculating the forces (called *ponderomotive* forces) which act on a dielectric in an arbitrary non-uniform electric field is fairly complicated and requires separate consideration for fluids (liquids or gases) and for solids. We shall take first the simpler case, that of fluid dielectrics. We denote by  $\mathbf{f}dV$  the force on a volume element  $dV$ , and call the vector  $\mathbf{f}$  the *force density*.

It is well known that the forces acting on any finite volume in a body can be reduced to forces applied to the surface of that volume (see TE, §2). This is a consequence of the law of conservation of momentum. The force acting on the matter in a volume  $dV$  is the change in its momentum per unit time. This change must be equal to the amount of momentum entering the volume through its surface per unit time. If we denote the momentum flux

† The change proportional to the square of the field is meant. It may be recalled that, in pyroelectric bodies, the change in the free energy contains also a term linear in the field, which is of no interest here.

tensor by  $-\sigma_{ik}$ , then†

$$\int f_i dV = \oint \sigma_{ik} df_k, \quad (15.1)$$

where the integration on the right is over the surface of the volume  $V$ . The tensor  $\sigma_{ik}$  is called the *stress tensor*. It is evident that  $\sigma_{ik} df_k = \sigma_{ik} n_k df$  is the  $i$ th component of the force on a surface element  $df$  ( $\mathbf{n}$  being a unit vector along the normal to the surface outwards from the volume under consideration).

Similarly, the total torque acting on a given volume also reduces to a surface integral, by virtue of the law of conservation of angular momentum. This reduction is possible because of the symmetry of the stress tensor ( $\sigma_{ik} = \sigma_{ki}$ ), which thus expresses the conservation law mentioned.

On transforming the surface integral in (15.1) into a volume integral, we obtain  $\int f_i dV = \int (\partial \sigma_{ik} / \partial x_k) dV$ , whence, since the volume of integration is arbitrary,

$$f_i = \partial \sigma_{ik} / \partial x_k. \quad (15.2)$$

This is a well-known formula giving the body forces in terms of the stress tensor.

Let us now calculate the stress tensor. Any small region of the surface may be regarded as plane, and the properties of the body and the electric field near it as uniform. Hence, to simplify the derivation, we can with no loss of generality consider a plane-parallel layer of material (with thickness  $h$  and uniform composition, density and temperature) in an electric field which is uniform.‡ This field may be imagined to be due to conducting planes, such as the plates of a capacitor, applied to the surfaces of the layer.

Following the general method for determining forces, we subject one of the plates (the upper one, say) to a virtual translation over an infinitesimal distance  $\xi$ , whose direction is arbitrary and need not be that of the normal  $\mathbf{n}$ . We shall suppose that the potential of the conductor remains unchanged at every point, and that the homogeneous deformation of the dielectric layer, resulting from the translation, is isothermal.

A force  $-\sigma_{ik} n_k$  is exerted by the layer on unit area of the surface. In the virtual displacement this force does work  $-\sigma_{ik} n_k \xi_i$ . The work done in an isothermal deformation at constant potential is equal to the decrease in  $\int \tilde{F} dV$ , i.e. in  $h\tilde{F}$  per unit surface area. Thus

$$\sigma_{ik} \xi_i n_k = \delta(h\tilde{F}) = h\delta\tilde{F} + \tilde{F}\delta h. \quad (15.3)$$

The thermodynamic quantities for the fluid depend (for given temperature and field) only on its density; deformations which do not change the density (i.e. pure shears) do not affect the thermodynamic state. We can therefore write for an isothermal variation  $\delta\tilde{F}$  in a fluid

$$\begin{aligned} \delta\tilde{F} &= \left( \frac{\partial \tilde{F}}{\partial \mathbf{E}} \right)_{T, \rho} \cdot \delta \mathbf{E} + \left( \frac{\partial \tilde{F}}{\partial \rho} \right)_{\mathbf{E}, T} \delta \rho \\ &= -\frac{\mathbf{D} \cdot \delta \mathbf{E}}{4\pi} + \left( \frac{\partial \tilde{F}}{\partial \rho} \right)_{\mathbf{E}, T} \delta \rho. \end{aligned} \quad (15.4)$$

† The force component  $f_i$  is not to be confused with the surface area component  $df_k$ .

‡ We thus ignore any terms in the stress tensor depending on the gradients of temperature, field, etc. These terms, however, are vanishingly small in comparison with terms which do not contain derivatives, in the same way as any terms containing derivatives which might appear in the relation between  $\mathbf{D}$  and  $\mathbf{E}$ .

The change in the density of the layer is related to the change in its thickness by  $\delta\rho = -\rho\delta h/h$ . The variation of the field is calculated as follows. At a given point in space (with position vector  $\mathbf{r}$ ) there appears matter which was originally at  $\mathbf{r} - \mathbf{u}$ , where  $\mathbf{u}$  is the particle displacement vector in the layer. Since, under the conditions stated (homogeneous deformation, and constant potential on the plates), each particle carries its potential with it, the change in the potential at a given point in space is  $\delta\phi = \phi(\mathbf{r} - \mathbf{u}) - \phi(\mathbf{r}) = -\mathbf{u} \cdot \text{grad } \phi = \mathbf{u} \cdot \mathbf{E}$ , where  $\mathbf{E}$  is the uniform field in the undeformed layer. Since the deformation is homogeneous, however, we have

$$\mathbf{u} = z\xi/h, \quad (15.5)$$

where  $z$  is the distance from the lower surface. Hence the variation of the field is

$$\delta\mathbf{E} = -\mathbf{n}(\mathbf{E} \cdot \xi)/h. \quad (15.6)$$

Substituting the above expressions in (15.4) and using also the fact that  $\delta h = \xi_z = \xi \cdot \mathbf{n}$ , we obtain

$$\begin{aligned} \sigma_{ik}\xi_i n_k &= \frac{1}{4\pi}(\mathbf{n} \cdot \mathbf{D})(\xi \cdot \mathbf{E}) - \xi \cdot \mathbf{n} \rho \frac{\partial \tilde{F}}{\partial \rho} + \xi \cdot \mathbf{n} \tilde{F} \\ &= \left\{ \frac{E_i D_k}{4\pi} - \rho \frac{\partial \tilde{F}}{\partial \rho} \delta_{ik} + \tilde{F} \delta_{ik} \right\} \xi_i n_k. \end{aligned}$$

Hence we have finally the following expression for the stress tensor:

$$\sigma_{ik} = [\tilde{F} - \rho(\partial \tilde{F} / \partial \rho)_{\mathbf{E}, T}] \delta_{ik} + E_i D_k / 4\pi. \quad (15.7)$$

In isotropic media, which are those here considered,  $\mathbf{E}$  and  $\mathbf{D}$  are parallel. Hence  $E_i D_k = E_k D_i$ , and the tensor (15.7) is symmetrical, as it should be.†

If the linear relation  $\mathbf{D} = \epsilon \mathbf{E}$  holds, then

$$\tilde{F} = F_0(\rho, T) - \epsilon E^2 / 8\pi; \quad (15.8)$$

see (10.17).  $F_0$  is the free energy per unit volume in the absence of the field. According to a well-known thermodynamic relation, the derivative of the free energy per unit mass with respect to the specific volume is the pressure:

$$\left[ \frac{\partial}{\partial(1/\rho)} \left( \frac{F_0}{\rho} \right) \right]_T = F_0 - \rho \left( \frac{\partial F_0}{\partial \rho} \right)_T = -P_0;$$

$P_0 = P_0(\rho, T)$  is the pressure which would be found in the medium in the absence of a field and for given values of  $\rho$  and  $T$ . Hence, substituting (15.8) in (15.7), we have

$$\sigma_{ik} = -P_0(\rho, T) \delta_{ik} - \frac{E^2}{8\pi} \left[ \epsilon - \rho \left( \frac{\partial \epsilon}{\partial \rho} \right)_T \right] \delta_{ik} + \frac{\epsilon E_i E_k}{4\pi}. \quad (15.9)$$

In a vacuum, this expression becomes the familiar Maxwell stress tensor of the electric field.‡

† It is not important that in this derivation  $\mathbf{E}$  is parallel to  $\mathbf{n}$ , since  $\sigma_{ik}$  can obviously depend only on the direction of  $\mathbf{E}$ , not on that of  $\mathbf{n}$ .

‡ See the first footnote to §5.



The forces exerted on the surface of separation by two adjoining media must be equal and opposite:  $\sigma_{ik}n_k = -\sigma'_{ik}n'_k$ , where the quantities with and without the prime refer to the two media. The normal vectors  $\mathbf{n}$  and  $\mathbf{n}'$  are in opposite directions, so that

$$\sigma_{ik}n_k = \sigma'_{ik}n_k. \quad (15.10)$$

At the boundary of two isotropic media the condition of equality of the tangential forces is satisfied identically. For, substituting (15.7) in (15.10) and taking the tangential component, we obtain  $\mathbf{E}_t D_n = \mathbf{E}'_t D'_n$ . This equation is satisfied by virtue of the boundary conditions of continuity on  $\mathbf{E}_t$  and  $D_n$ . The condition of equality of the normal forces is, however, a non-trivial condition on the pressure difference between the two media.

For example, let us consider a boundary between a liquid and the atmosphere (for which we can put  $\varepsilon = 1$ ). Denoting by a prime quantities pertaining to the atmosphere, and using formula (15.9) for  $\sigma_{ik}$ , we have

$$-P_0(\rho, T) + \frac{E^2}{8\pi} \rho \left( \frac{\partial \varepsilon}{\partial \rho} \right)_T + \frac{\varepsilon}{8\pi} (E_n^2 - E_t^2) = -P_{\text{atm}} + \frac{1}{8\pi} (E_n'^2 - E_t'^2).$$

Using the boundary conditions  $E_t = E'_t$ ,  $D_n = \varepsilon E_n = D'_n = E'_n$ , we can rewrite this equation as

$$P_0(\rho, T) - P_{\text{atm}} = \frac{\rho E^2}{8\pi} \left( \frac{\partial \varepsilon}{\partial \rho} \right)_T - \frac{\varepsilon - 1}{8\pi} (\varepsilon E_n^2 + E_t^2). \quad (15.11)$$

This relation is to be taken as determining the density  $\rho$  of the liquid near its surface from the electric field in it.

Let us now determine the body forces acting in a dielectric medium. Differentiating (15.9) in accordance with (15.2) gives

$$f_i = \frac{\partial}{\partial x_i} \left[ -P_0 + \frac{E^2}{8\pi} \rho \left( \frac{\partial \varepsilon}{\partial \rho} \right)_T \right] - \frac{E^2}{8\pi} \frac{\partial \varepsilon}{\partial x_i} + \frac{1}{4\pi} \left[ -\frac{1}{2} \varepsilon \frac{\partial}{\partial x_i} E^2 + \frac{\partial}{\partial x_k} (E_i D_k) \right].$$

On using the equation  $\text{div } \mathbf{D} \equiv \partial D_k / \partial x_k = 0$ , the expression in the brackets in the last term can be reduced to

$$-\varepsilon E_k \frac{\partial E_k}{\partial x_i} + D_k \frac{\partial E_i}{\partial x_k} = -D_k \left( \frac{\partial E_k}{\partial x_i} - \frac{\partial E_i}{\partial x_k} \right),$$

which is zero, since  $\text{curl } \mathbf{E} = 0$ . Thus we have

$$\mathbf{f} = -\text{grad } P_0(\rho, T) + \frac{1}{8\pi} \text{grad} \left[ E^2 \rho \left( \frac{\partial \varepsilon}{\partial \rho} \right)_T \right] - \frac{E^2}{8\pi} \text{grad } \varepsilon \quad (15.12)$$

(H. Helmholtz, 1881).

If the dielectric contains extraneous charges with density  $\rho_{\text{ex}}$ , the force  $\mathbf{f}$  contains a further term  $\mathbf{E} \text{ div } \mathbf{D} / 4\pi$ , or, since  $\text{div } \mathbf{D} = 4\pi \rho_{\text{ex}}$ ,

$$\rho_{\text{ex}} \mathbf{E}; \quad (15.13)$$

however, it should not be supposed that this result is obvious (cf. §16, Problem 3).

In a gas, as already mentioned in §7, we can assume the difference  $\varepsilon - 1$  to be proportional to the density. Then  $\rho \partial \varepsilon / \partial \rho = \varepsilon - 1$ , and formula (15.12) takes the simpler form

$$\mathbf{f} = -\text{grad } P_0 + \frac{\varepsilon - 1}{8\pi} \text{grad } E^2. \quad (15.14)$$



Formula (15.12) is valid for media of both uniform and non-uniform composition. In the latter case  $\varepsilon$  is a function not only of  $\rho$  and  $T$  but also of the concentration of the mixture, which varies through the medium. In a body of uniform composition, on the other hand,  $\varepsilon$  is a function only of  $\rho$  and  $T$ , and  $\mathbf{grad} \varepsilon$  can be written as

$$\mathbf{grad} \varepsilon = (\partial \varepsilon / \partial T)_\rho \mathbf{grad} T + (\partial \varepsilon / \partial \rho)_T \mathbf{grad} \rho.$$

Then (15.12) becomes

$$\mathbf{f} = -\mathbf{grad} P_0(\rho, T) + \frac{\rho}{8\pi} \mathbf{grad} \left[ E^2 \left( \frac{\partial \varepsilon}{\partial \rho} \right)_T \right] - \frac{E^2}{8\pi} \left( \frac{\partial \varepsilon}{\partial T} \right)_\rho \mathbf{grad} T. \quad (15.15)$$

If the temperature also is constant throughout the body, the third term on the right is zero, and in the first term  $\mathbf{grad} P_0$  can be replaced by  $\rho \mathbf{grad} \zeta_0$ , in accordance with the thermodynamic identity for the chemical potential in the absence of a field,  $\rho d\zeta_0 = dP_0 - S_0 dT$ . Thus

$$\begin{aligned} \mathbf{f} &= -\rho \mathbf{grad} \left[ \zeta_0 - \frac{E^2}{8\pi} \left( \frac{\partial \varepsilon}{\partial \rho} \right)_T \right] \\ &= -\rho \mathbf{grad} \zeta, \end{aligned} \quad (15.16)$$

where  $\zeta$  is the chemical potential in an electric field (see (10.19)).

In particular, the condition of mechanical equilibrium  $\mathbf{f} = 0$  is, for constant temperature,

$$\zeta = \zeta_0 - (E^2/8\pi)(\partial \varepsilon / \partial \rho)_T = \text{constant}, \quad (15.17)$$

in accordance with the thermodynamic condition of equilibrium. This condition can usually be written still more simply. The change in density of the medium due to the field is proportional to  $E^2$ . Hence, if the medium is of uniform density in the absence of the field, we can put  $\rho = \text{constant}$  in the last two terms in (15.15) when the field is present; an allowance for the change in  $\rho$  is beyond the accuracy of formulae which assume the linear relation  $\mathbf{D} = \varepsilon \mathbf{E}$ . Then, equating to zero  $\mathbf{f}$  from (15.15), we obtain the equilibrium condition at constant temperature in the form

$$P_0(\rho, T) - (\rho E^2/8\pi)(\partial \varepsilon / \partial \rho)_T = \text{constant}, \quad (15.18)$$

which differs from (15.17) in that  $\zeta_0$  is replaced by  $P_0/\rho$ .

To close this section, we shall show how the expression (15.12) for the force may be derived directly from (14.1) if the calculation of the stress tensor is not required.

Let us consider an infinite inhomogeneous dielectric medium subjected to a small isothermal deformation that is zero at infinity. The variation  $\delta \varepsilon$  is made up of two parts: (1) the change

$$\varepsilon(\mathbf{r} - \mathbf{u}) - \varepsilon(\mathbf{r}) = -\mathbf{u} \cdot \mathbf{grad} \varepsilon$$

due to the fact that a particle is brought by the deformation from  $\mathbf{r} - \mathbf{u}$  to  $\mathbf{r}$ , and (2) the change  $-(\partial \varepsilon / \partial \rho)_T \rho \operatorname{div} \mathbf{u}$ , due to the change in the density of the substance at the point  $\mathbf{r}$ ; it is known (see *TE*, §1) that  $\operatorname{div} \mathbf{u}$  is the relative change in the volume element, so that the change in the density is  $\delta \rho = -\rho \operatorname{div} \mathbf{u}$ . The variation in the free energy is therefore

$$\begin{aligned} \delta \mathcal{F} &= \delta \mathcal{F}_0 - \int \delta \varepsilon (E^2/8\pi) dV \\ &= - \int P_0 \operatorname{div} \mathbf{u} dV + \int (E^2/8\pi) [\mathbf{u} \cdot \mathbf{grad} \varepsilon + (\partial \varepsilon / \partial \rho)_T \rho \operatorname{div} \mathbf{u}] dV; \end{aligned} \quad (15.19)$$

the first term is the variation in the free energy when the field is absent. Integrating by parts the  $\text{div } \mathbf{u}$  terms in (15.19) and comparing the result with the expression  $\delta \mathcal{F} = - \int \mathbf{u} \cdot \mathbf{f} dV$  for the free energy variation in terms of the work done by the forces  $\mathbf{f}$ , we arrive at (15.12).

### §16. Electric forces in solids

The dielectric properties of a solid body change not only when its density changes (as with liquids) but also under deformations (pure shears) which do not affect the density. Let us first consider bodies which are isotropic in the absence of the field. In general, the deformed body is no longer isotropic; in consequence, its dielectric properties also become anisotropic, and the scalar permittivity  $\epsilon$  is replaced by the dielectric tensor  $\epsilon_{ik}$ .

The state of a slightly deformed body is described by the strain tensor

$$u_{ik} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right),$$

where  $\mathbf{u}(x, y, z)$  is the displacement vector for points in the body. Since these quantities are small, only the first-order terms in  $u_{ik}$  need be retained in the variation of the components  $\epsilon_{ik}$ . Accordingly, we represent the dielectric tensor of the deformed body as

$$\epsilon_{ik} = \epsilon_0 \delta_{ik} + a_1 u_{ik} + a_2 u_{ll} \delta_{ik}. \quad (16.1)$$

Here  $\epsilon_0$  is the permittivity of the undeformed body, and the other two terms, which contain the scalar constants  $a_1, a_2$ , form the most general tensor of rank two which can be constructed linearly from the components  $u_{ik}$ .

Let us now see where the derivation given in §15 must be modified. Since, in a solid body,  $\tilde{F}$  depends on all the components of the strain tensor, we must replace (15.4) by  $\delta \tilde{F} = -\mathbf{D} \cdot \delta \mathbf{E} / 4\pi + (\partial \tilde{F} / \partial u_{ik}) \delta u_{ik}$ . For the virtual displacement considered, the vector  $\mathbf{u}$  is given by formula (15.5), so that the strain tensor is  $u_{ik} = (\xi_i n_k + \xi_k n_i) / 2h$ . Substituting this in  $\delta \tilde{F}$  and using the symmetry of the tensor  $u_{ik}$ , and therefore of the derivatives  $\partial \tilde{F} / \partial u_{ik}$ , we obtain

$$\delta \tilde{F} = -\mathbf{D} \cdot \delta \mathbf{E} / 4\pi + (\xi_i n_k / h) \partial \tilde{F} / \partial u_{ik}. \quad (16.2)$$

It is now evident that we find, instead of (15.7), the following expression for the stress tensor:†

$$\sigma_{ik} = \tilde{F} \delta_{ik} = (\partial \tilde{F} / \partial u_{ik})_{T, E} + E_i D_k / 4\pi. \quad (16.3)$$

Formula (16.3) is valid whatever the relation between  $\mathbf{D}$  and  $\mathbf{E}$ . For a body which is neither pyroelectric nor piezoelectric, so that  $D_i = \epsilon_{ik} E_k$ ,  $\tilde{F}$  is given by formula (13.4) and the required derivatives are  $\partial \tilde{F} / \partial u_{ik} = \partial F_0 / \partial u_{ik} - (a_1 E_i E_k + a_2 E^2 \delta_{ik}) / 8\pi$ . We then put  $\epsilon_{ik} = \epsilon_0 \delta_{ik}$  everywhere in (16.3) and obtain the following formula for the stress tensor:

$$\sigma_{ik} = \sigma_{ik}^{(0)} + (2\epsilon_0 - a_1) E_i E_k / 8\pi - (\epsilon_0 + a_2) E^2 \delta_{ik} / 8\pi. \quad (16.4)$$

† The quantity  $\tilde{F}$  in this formula, and in all preceding formulae, is the free energy per unit volume. In the theory of elasticity, however, a somewhat different definition is usual: the thermodynamic quantities are referred to the amount of matter contained in unit volume of the undeformed body, which may after deformation occupy some other volume. It is easy to go from one definition to the other by expressing the relative volume change in the deformation in terms of the tensor  $u_{ik}$ ; on account of the presence of the derivative with respect to  $u_{ik}$  in (16.3), this must be done with allowance for second-order terms. As a result, the first two terms on the right of (16.3) combine into one of the form  $\partial \tilde{F} / \partial u_{ik}$ , in accordance with the usual formula of elasticity theory.

$\sigma^{(0)}_{ik}$  is the stress tensor in the absence of an electric field, determined by the moduli of rigidity and compression according to the ordinary formulae of the theory of elasticity.

Let us now make similar calculations for anisotropic solids.† The necessary modification of the above argument is as follows. When the layer undergoes a virtual deformation, its crystallographic axes are rotated, and their orientation relative to the electric field is therefore changed. On account of the anisotropy of the dielectric properties of the crystal, this leads to an additional change in  $\tilde{F}$  not shown in (16.2). To calculate this change we can equally well suppose that the crystal axes rotate through some angle  $\delta\phi$  relative to the field  $\mathbf{E}$ , or that the field rotates through an angle  $-\delta\phi$  relative to the axes, and the latter approach is the more convenient.

Thus the variation of the field (15.6) considered above must be augmented by the change in  $\mathbf{E}$  on rotation through an angle  $-\delta\phi$ :

$$\delta\mathbf{E} = -\mathbf{n}(\mathbf{E} \cdot \boldsymbol{\xi})/h - \delta\phi \times \mathbf{E}.$$

The angle  $\delta\phi$  is related to the displacement vector  $\mathbf{u}$  in the deformation by  $\delta\phi = \frac{1}{2}\text{curl } \mathbf{u}$ ; this equation is easily obtained by noticing that, when the body rotates through an angle  $\delta\phi$ , its points are displaced by  $\mathbf{u} = \delta\phi \times \mathbf{r}$ . Substituting  $\mathbf{u}$  from (15.5), we find  $\delta\phi = \text{curl } \mathbf{z}\xi/2h = \mathbf{n} \times \boldsymbol{\xi}/2h$ , and  $\delta\mathbf{E} = -\mathbf{n}(\mathbf{E} \cdot \boldsymbol{\xi})/h + \mathbf{E} \times (\mathbf{n} \times \boldsymbol{\xi})/2h = -[\mathbf{n}(\mathbf{E} \cdot \boldsymbol{\xi}) + \boldsymbol{\xi}(\mathbf{n} \cdot \mathbf{E})]/2h$ . The first term in (16.2) becomes

$$-\frac{1}{4\pi} \mathbf{D} \cdot \delta\mathbf{E} = \frac{1}{8\pi h} [(\mathbf{n} \cdot \mathbf{D})(\boldsymbol{\xi} \cdot \mathbf{E}) + (\boldsymbol{\xi} \cdot \mathbf{D})(\mathbf{n} \cdot \mathbf{E})] = \frac{1}{4\pi h} \xi_i n_k \cdot \frac{1}{2}(E_i D_k + E_k D_i).$$

Hence we see that the product  $E_i D_k$  in (16.3) must be replaced by the second factor in the last expression:

$$\sigma_{ik} = \tilde{F} \delta_{ik} + \frac{\partial \tilde{F}}{\partial u_{ik}} + \frac{1}{8\pi} (E_i D_k + E_k D_i). \quad (16.5)$$

This expression is symmetrical in the suffixes  $i$  and  $k$ , as it should be.

The expression (16.1) for the dielectric tensor, involving two scalar constants, must be replaced in the case of a deformed crystal by

$$\epsilon_{ik} = \epsilon^{(0)}_{ik} + a_{iklm} u_{lm}, \quad (16.6)$$

where  $a_{iklm}$  is a constant tensor of rank four, symmetrical with respect to the pairs of suffixes  $i, k$  and  $l, m$  (but not with respect to an interchange of these pairs). The number of independent non-zero components of this tensor depends on the crystal class.

We shall not pause to write out here the formula for the stress tensor (analogous to (16.4)) which is obtained by using (16.6).

The formulae which we have obtained give the stresses inside a solid dielectric. They are not needed, however, if we wish to determine the total force  $\mathbf{F}$  or the total torque  $\mathbf{K}$  exerted on the body by the external field. Let us consider a body immersed in a fluid medium and kept at rest there. The total force on it is equal to the integral  $\oint \sigma_{ik} n_k df$ , taken over the surface. Since the force  $\sigma_{ik} n_k$  is continuous, it does not matter whether this integral is calculated from the values of  $\sigma_{ik}$  given by (16.4) or from formula (15.9), which relates to the

† We shall see in §17 that the phenomenon of electrostriction in crystals may, for some types of symmetry, differ markedly from that in isotropic bodies. Such crystals are said to be *piezoelectric*. Here, however, we discuss only electrostriction in non-piezoelectric crystals.

medium surrounding the body. Let us suppose that this medium is in mechanical and thermal equilibrium. Then the calculation is further simplified if we use the condition of equilibrium (15.18). From this condition, part of the stress tensor (15.9) is constant through the body, being a uniform compressing or expanding pressure and making no contribution to the total force  $\mathbf{F}$  and torque  $\mathbf{K}$  acting on the body. These can therefore be calculated by writing  $\sigma_{ik}$  as

$$\sigma_{ik} = (\epsilon/4\pi)(E_i E_k - \frac{1}{2}E^2 \delta_{ik}) \quad (16.7)$$

simply, where  $\mathbf{E}$  is the field in the fluid and  $\epsilon$  its permittivity; this expression differs only by a factor  $\epsilon$  from the Maxwell stress tensor of the electric field in a vacuum. Thus

$$\mathbf{F} = (\epsilon/4\pi) \oint [\mathbf{E}(\mathbf{n} \cdot \mathbf{E}) - \frac{1}{2}E^2 \mathbf{n}] d\mathbf{f}, \quad (16.8)$$

$$\mathbf{K} = (\epsilon/4\pi) \oint [\mathbf{r} \times \mathbf{E}(\mathbf{n} \cdot \mathbf{E}) - \frac{1}{2}E^2 \mathbf{r} \times \mathbf{n}] d\mathbf{f}. \quad (16.9)$$

It may also be noted that, since the fluid is in equilibrium, we can take these integrals over any closed surface which surrounds the body in question (but, of course, does not enclose any of the charged bodies which are sources of the field).

The calculation of the total force on a dielectric in an electric field in a vacuum can also be approached in another way by expressing this force, not in terms of the actual field, but in terms of the field  $\mathcal{E}$  which would be produced by the given sources in the absence of the dielectric; this is the "external field" in which the body is placed. Here it is assumed that the distribution of charges producing the field is unchanged when the body is brought in. This condition may not be fulfilled in practice—for example, if the charges are distributed over the surface of an extended conductor and the dielectric is brought to a finite distance from it.

In a virtual translation of the body over an infinitesimal distance  $\mathbf{u}$ , the total free energy of the body varies, according to (11.3), by  $\delta\mathcal{F} = -\int \mathbf{P} \cdot \delta\mathcal{E} dV$ , where  $\delta\mathcal{E} = \mathcal{E}(\mathbf{r} + \mathbf{u}) - \mathcal{E}(\mathbf{r}) = (\mathbf{u} \cdot \text{grad})\mathcal{E}$  is the change in the field at any given point in the body. Since  $\mathbf{u}$  = constant and  $\text{curl } \mathcal{E} = 0$ , we have  $\mathbf{P} \cdot (\mathbf{u} \cdot \text{grad})\mathcal{E} = \mathbf{P} \cdot \text{grad}(\mathbf{u} \cdot \mathcal{E}) = \mathbf{u} \cdot (\mathbf{P} \cdot \text{grad})\mathcal{E}$ , so that

$$\delta\mathcal{F} = -\mathbf{u} \cdot \int (\mathbf{P} \cdot \text{grad})\mathcal{E} dV.$$

But  $\delta\mathcal{F} = -\mathbf{u} \cdot \mathbf{F}$ , and we therefore have for the required force†

$$\mathbf{F} = \int (\mathbf{P} \cdot \text{grad})\mathcal{E} dV. \quad (16.10)$$

Similarly, the total torque on the body can be determined. We shall not go through the calculation, but merely give the result:

$$\mathbf{K} = \int \mathbf{P} \times \mathcal{E} dV + \int \mathbf{r} \times (\mathbf{P} \cdot \text{grad})\mathcal{E} dV. \quad (16.11)$$

In an almost uniform field, which may be regarded as constant over the dimensions of the body, formula (16.10) gives to a first approximation

$$\mathbf{F} = \left( \int \mathbf{P} dV \cdot \text{grad} \right) \mathcal{E} = (\mathcal{P} \cdot \text{grad}) \mathcal{E}, \quad (16.12)$$

† It should be emphasized, however, that the integrand in (16.10) cannot be interpreted as the force density. The reason is that the local forces in the dielectric arise not only from the field  $\mathcal{E}$  but also from the internal fields which, by Newton's third law, contribute nothing to the total force, though they modify the distribution of forces over the volume of the body.

where  $\mathcal{P}$  is the total dipole moment of the polarized dielectric; this result, of course, could have been obtained by direct differentiation of  $\mathcal{F}$  from (11.8). In formula (16.11) we neglect the second term in the first approximation and reach the natural conclusion that

$$\mathbf{K} = \mathcal{P} \times \mathcal{E}. \quad (16.13)$$

## PROBLEMS

**PROBLEM 1.** A dielectric sphere with radius  $a$  in a uniform external field  $\mathcal{E}$  is cut in half by a plane perpendicular to the field. Determine the force of attraction between the hemispheres.

**SOLUTION.** We imagine the hemispheres separated by an infinitely narrow slit and determine the force from formula (16.8) with  $\varepsilon = 1$ , integrating over the surface of a hemisphere;  $\mathbf{E}$  is the field in the vacuum near the surface. According to (8.2) the field  $\mathbf{E}^{(i)}$  inside the sphere is uniform and equal to  $3\mathcal{E}/(2 + \varepsilon)$ , where  $\varepsilon$  is the permittivity of the sphere. The field in the slit is perpendicular to the surface and is  $\mathbf{E} = \mathbf{D}^{(i)} = 3\varepsilon\mathcal{E}/(2 + \varepsilon)$ . On the outer surface of the sphere we have

$$E_r = D_r^{(i)} = \frac{3\varepsilon}{2 + \varepsilon} \mathcal{E} \cos \theta, \quad E_\theta = E_\theta^{(i)} = -\frac{3}{2 + \varepsilon} \mathcal{E} \sin \theta,$$

where  $\theta$  is the angle between the position vector and the direction of  $\mathcal{E}$ . A calculation of the integral gives an attractive force†

$$F = 9(\varepsilon - 1)^2 a^2 \mathcal{E}^2 / 16(\varepsilon + 2)^2.$$

**PROBLEM 2.** Determine the change in shape of a dielectric sphere in a uniform external electric field.

**SOLUTION.** As in §5, Problem 4. In determining the change in shape, we assume the volume of the sphere to be unchanged.‡ The elastic part of the free energy is given by the same expression as in §5, Problem 4. The electric part is given by (8.9):

$$-\frac{1}{2} \mathcal{P} \cdot \mathcal{E} = -\frac{V}{8\pi} \frac{\varepsilon^{(x)} - 1}{1 + n(\varepsilon^{(x)} - 1)} \mathcal{E}^2.$$

and the permittivity in the  $x$ -direction is, by (16.1),  $\varepsilon^{(x)} = \varepsilon_0 + a_1 u_{xx} = \varepsilon_0 + \frac{2}{3}a_1(u_{xx} - u_{yy}) = \varepsilon_0 + \frac{2}{3}a_1(a - b)/R$ . From the condition that the total free energy be a minimum we find (since the quantity concerned is small)

$$\frac{a - b}{R} = \frac{9\mathcal{E}^2}{40\pi\mu} \frac{(\varepsilon_0 - 1)^2 + 5a_1/2}{(\varepsilon_0 + 2)^2}.$$

For  $\varepsilon_0 \rightarrow \infty$  this tends to the value for a conducting sphere.

**PROBLEM 3.** Determine the body forces in an isotropic solid dielectric, assumed homogeneous, when extraneous charges are present in it.

**SOLUTION.** Assuming  $\varepsilon_0, a_1, a_2$  constant and using the equations  $\text{curl } \mathbf{E} = 0, \text{div } \mathbf{D} \cong \varepsilon_0 \text{div } \mathbf{E} = 4\pi\rho_{\text{ex}}$ , we have from (16.4)

$$f_i = \frac{\partial \sigma_{ik}}{\partial x_k} = \frac{\partial \sigma_{ik}^{(0)}}{\partial x_k} - \frac{1}{8\pi} (\frac{1}{2}a_1 + a_2) \frac{\partial E^2}{\partial x_i} + \left(1 - \frac{a_1}{2\varepsilon_0}\right) \rho_{\text{ex}} E_i.$$

## §17. Piezoelectrics

The internal stresses which occur in an isotropic dielectric in an electric field are proportional to the square of the field. The effect is similar in crystals belonging to some of

† It is by chance that, in the limit  $\varepsilon \rightarrow \infty$ , this expression tends to the result obtained in §5, Problem 3, for a conducting sphere (indeed, the forces are in opposite directions). The two cases are evidently not physically equivalent, because there is no field in the slit between two conducting hemispheres at the same potential, whereas in this problem there is a field in the slit.

‡ The change in volume is determined in §12, Problem 1.

the crystal classes. For certain types of symmetry, however, the electrostriction properties of the crystals are quite different. The internal stresses in these *piezoelectric* bodies resulting from an electric field are proportional to the field itself. The converse effect also occurs: the deformation of a piezoelectric is accompanied by the appearance in it of a field proportional to the deformation.

Since in a piezoelectric only the principal (linear) effect is of interest, we can neglect the terms quadratic in the field in the general formula (16.5). Then  $\sigma_{ik} = \tilde{F}\delta_{ik} + (\partial\tilde{F}/\partial u_{ik})_{T,E}$ . In this section we shall use the thermodynamic quantities referred to the matter in unit volume of the undeformed body (see the first footnote to §16). Taking  $\tilde{F}$  in this sense, we have simply

$$\sigma_{ik} = (\partial\tilde{F}/\partial u_{ik})_{T,E}. \quad (17.1)$$

Accordingly, the thermodynamic relation for the differential  $d\tilde{F}$  is

$$d\tilde{F} = -S dT + \sigma_{ik} du_{ik} - \mathbf{D} \cdot d\mathbf{E}/4\pi. \quad (17.2)$$

The following remark should be made concerning the last term. In the form given here, this term (taken from (10.9)) pertains, strictly speaking, to unit volume of the deformed body. By ignoring this fact, we commit an error which, in the case of a piezoelectric, is of a higher order of smallness than the remaining terms in (17.2).

The independent variables in (17.2) include the components of the tensor  $u_{ik}$ . It is sometimes convenient to use instead the components  $\sigma_{ik}$ . To do so, we must introduce the thermodynamic potential, defined as

$$\tilde{\Phi} = \tilde{F} - u_{ik}\sigma_{ik}. \quad (17.3)$$

For the differential of this quantity we have

$$d\tilde{\Phi} = -S dT - u_{ik} d\sigma_{ik} - \mathbf{D} \cdot d\mathbf{E}/4\pi. \quad (17.4)$$

It must be emphasized that the use of the thermodynamic potential  $\tilde{\Phi}$  in electrodynamics in accordance with formulae (17.3) and (17.4) rests on the validity of (17.1) and so is possible only for piezoelectric bodies.

Having thus defined the necessary thermodynamic quantities, let us now ascertain the piezoelectric properties of crystals. If  $\sigma_{ik}$  and  $E_k$  are taken as independent variables, the induction  $\mathbf{D}$  must be regarded as a function of them, and an expansion of this function must retain the terms linear in them. The linear terms in the expansion of the components of a vector in powers of the components of a tensor of rank two can be written, in the most general case, as  $4\pi\gamma_{i,kl}\sigma_{kl}$ , where the constants  $\gamma_{i,kl}$  form a tensor of rank three, and the factor  $4\pi$  is introduced for convenience. Since the tensor  $\sigma_{kl}$  is symmetrical, it is clear that the tensor  $\gamma_{i,kl}$  may also be supposed to have the symmetry property

$$\gamma_{i,kl} = \gamma_{i,lk}. \quad (17.5)$$

For clarity we separate the symmetrical suffixes from the remaining one by a comma. We call  $\gamma_{i,kl}$  the *piezoelectric tensor*. If it is known, the piezoelectric properties of the crystal are entirely determined.

Adding the piezoelectric terms to the expression (13.1) for the electric induction in the crystal, we have

$$D_i = D_{0i} + \epsilon_{ik}E_k + 4\pi\gamma_{i,kl}\sigma_{kl}. \quad (17.6)$$

Corresponding additional terms appear in the thermodynamic quantities. The thermo-

dynamic potential of a non-piezoelectric crystal in the absence of a field is  $\tilde{\Phi} = \Phi = \Phi_0 - \frac{1}{2}\mu_{iklm}\sigma_{ik}\sigma_{lm}$ , where  $\Phi_0$  pertains to the undeformed body, and the second term is the ordinary elastic energy, determined by the *elastic constant tensor*  $\mu_{iklm}$ .† For a piezoelectric we have

$$\tilde{\Phi} = \Phi_0 - \frac{1}{2}\mu_{iklm}\sigma_{ik}\sigma_{lm} - \varepsilon_{ik}E_iE_k/8\pi - E_iD_{0i}/4\pi - \gamma_{i,kl}E_i\sigma_{kl}. \quad (17.7)$$

The form of the last three terms is given by the fact that the derivatives of  $\tilde{\Phi}$  with respect to  $E_i$  (for given temperature and internal stresses), found from the relation  $D_i = -4\pi\partial\tilde{\Phi}/\partial E_i$ , must accord with (17.6).

Knowing  $\tilde{\Phi}$ , we can obtain from (17.4) a formula giving the strain tensor in terms of the stresses  $\sigma_{ik}$  and the field  $\mathbf{E}$ :

$$u_{ik} = -(\partial\tilde{\Phi}/\partial\sigma_{ik})_{T,E} = \mu_{iklm}\sigma_{lm} + \gamma_{l,ik}E_l. \quad (17.8)$$

It should be mentioned that to regard the quantities  $\mu_{iklm}$  and  $\varepsilon_{ik}$  for a piezoelectric as elastic constants and permittivity is to some extent conventional. With the definitions used here, they give respectively the strains as functions of the elastic stresses for a given field, and the induction as a function of the field for given stresses. If, however, the deformation occurs with a given value of the induction, or we consider the induction as a function of the field for given strains, the elastic constants and the permittivity will be represented by other quantities, which can be expressed as somewhat complex functions of the components of the tensors  $\mu$ ,  $\varepsilon$  and  $\gamma$ .

The field in a piezoelectric body must be determined together with its deformation, leading to a problem in both electrostatics and elasticity theory. We must seek a simultaneous solution of the electrostatic equations

$$\operatorname{div} \mathbf{D} = 0, \quad \operatorname{curl} \mathbf{E} = 0, \quad (17.9)$$

with  $\mathbf{D}$  given by (17.6), and the equations of elastic equilibrium

$$\partial\sigma_{ik}/\partial x_k = 0, \quad (17.10)$$

with the appropriate boundary conditions at the surface of the body and use of the relation (17.8) between  $\sigma_{ik}$  and the strains. In general this problem is very complex.

The problem is much simplified for a body of ellipsoidal form with a free surface (i.e. one subject to no external mechanical forces). In this case (§8), the field inside the body is uniform; the deformation is therefore homogeneous, and the elastic stresses  $\sigma_{ik} = 0$ .

Finally, let us consider which types of crystal symmetry allow the existence of piezoelectricity; in other words, what are the restrictions imposed on the components of the tensor  $\gamma_{i,kl}$  by the symmetry conditions. In general, this tensor (which is symmetrical in

† The tensor  $\mu_{iklm}$  determines the relation between stress and strain:

$$u_{ik} = -\partial\Phi/\partial\sigma_{ik} = \mu_{iklm}\sigma_{lm}.$$

In TE, §10, the converse relation  $\sigma_{ik} = \lambda_{iklm}u_{lm}$  is used. It is evident that the symmetry properties of the tensor  $\mu_{iklm}$  are exactly the same as those of  $\lambda_{iklm}$ .

The free energy  $F$  contains the elastic energy with the plus sign:

$$F_{e1} = \frac{1}{2}\lambda_{iklm}u_{ik}u_{lm}.$$

The thermodynamic potential is obtained from  $F$  by subtracting  $\sigma_{ik}u_{ik}$ , and so

$$\Phi_{e1} = F_{e1} - \sigma_{ik}u_{ik} = -\frac{1}{2}\lambda_{iklm}u_{ik}u_{lm} = -\frac{1}{2}\mu_{iklm}\sigma_{ik}\sigma_{lm}.$$



the suffixes  $k$  and  $l$ ) has 18 independent non-zero components, but in reality the number of independent components is usually much smaller.

In all symmetry transformations of a given crystal, the components of the tensor  $\gamma_{i,kl}$  must remain unaltered in value. Hence it follows at once that no piezoelectric body can have a centre of symmetry or, in particular, be isotropic. For, on reflection in the centre (i.e. change of sign of all three coordinates), the components of a tensor of rank three change sign.

Of the 32 crystal classes, only 20 allow piezoelectricity. These comprise the ten enumerated in §13 as allowing pyroelectricity (all pyroelectrics are also piezoelectrics) and the ten following classes:

orthorhombic system:  $D_2$   
 tetragonal system:  $D_4, D_{2d}, S_4$   
 rhombohedral system:  $D_3$   
 hexagonal system:  $D_6, C_{3h}, D_{3h}$   
 cubic system:  $T, T_d$ .

The non-zero components of the piezoelectric tensor for each class are given in the Problems below.

Mention may also be made here of a phenomenon akin to piezoelectricity, which results from the "deformation" of a liquid crystal. We shall consider what are called *nematic* crystals (SP 1, §140), liquids in which there is a distinctive direction of preferred orientation of the molecules. At each point in the medium, this direction is specified by a unit vector  $\mathbf{d}$ , the *director* of the crystal. In an undeformed liquid crystal,  $\mathbf{d}$  has the same direction everywhere, but in a deformed one this direction is a function of the coordinates. The expansion (17.6) corresponds to an expression for the induction in a liquid crystal in the form

$$D_i = \epsilon_{ik} E_k + 4\pi e_1 d_i \operatorname{div} \mathbf{d} + 4\pi e_2 (\operatorname{curl} \mathbf{d} \times \mathbf{d})_i, \quad (17.11)$$

where  $e_1$  and  $e_2$  are scalar coefficients (R. B. Meyer, 1969).† The last two terms, which describe the effect in question, constitute the most general polar vector that can be formed from  $\mathbf{d}$  and its first derivatives with respect to the coordinates. The expression (17.11) is automatically invariant under a change in the sign of  $\mathbf{d}$ .

The permittivity tensor of a nematic crystal has the same symmetry as for uniaxial crystals, the axis of symmetry being represented by the local (at each point) direction of the director. The tensor  $\epsilon_{ik}$  may be expressed as

$$\epsilon_{ik} = \epsilon_0 \delta_{ik} + \epsilon_a d_i d_k, \quad (17.12)$$

with two independent constants  $\epsilon_0$  and  $\epsilon_a$ .

## PROBLEMS

**PROBLEM 1.** Determine the non-zero components of the tensor  $\gamma_{i,kl}$  for non-pyroelectric crystal classes which allow piezoelectricity.

**SOLUTION.** The class  $D_2$  has three mutually perpendicular twofold axes of symmetry, which we take as the axes of  $x$ ,  $y$  and  $z$ . Rotations through  $180^\circ$  about these axes change the sign of two out of the three coordinates.

† Pyroelectricity in nematic crystals is in practice unknown, and we therefore put  $\mathbf{D}_0 = 0$ .



Since the components  $\gamma_{i,kl}$  are transformed as the products  $x_i x_k x_l$ , the only non-zero components are those with three different suffixes:  $\gamma_{x,yz}, \gamma_{z,xy}, \gamma_{y,zx}$ . (The other non-zero components are equal to these, since  $\gamma_{i,kl} = \gamma_{i,lk}$ .) Accordingly, the piezoelectric part of the thermodynamic potential is†

$$\tilde{\Phi}_{\text{pie}} = -2(\gamma_{x,yz} E_x \sigma_{yz} + \gamma_{y,zx} E_y \sigma_{zx} + \gamma_{z,xy} E_z \sigma_{xy}). \quad (1)$$

The class  $D_{2d}$  is obtained by adding to the axes of class  $D_2$  two planes of symmetry passing through one axis (the  $z$ -axis, say) and bisecting the angles between the other two. Reflection in one of these planes gives the transformation  $x \rightarrow y, y \rightarrow x, z \rightarrow z$ . Hence the components  $\gamma_{i,kl}$  which differ by interchange of  $x$  and  $y$  must be equal, so that only two out of the three coefficients in (1) are now independent:  $\gamma_{z,xy}, \gamma_{x,yz} = \gamma_{y,zx}$ .

The class  $T$  is obtained from the class  $D_2$  by adding four diagonal threefold axes of symmetry, rotations about which effect a cyclic permutation of  $x, y, z$ , e.g.  $x \rightarrow z, y \rightarrow x, z \rightarrow y$ . Hence all three coefficients in (1) are equal:  $\gamma_{x,yz} = \gamma_{y,zx} = \gamma_{z,xy}$ . The same result is obtained for the cubic class  $T_d$ .

The class  $D_4$  has one fourfold axis of symmetry (the  $z$ -axis, say) and four twofold axes lying in the  $xy$ -plane. Here the symmetry elements of the class  $D_2$  are supplemented by a rotation through  $90^\circ$  about the  $z$ -axis, i.e. the transformation  $x \rightarrow y, y \rightarrow -x, z \rightarrow z$ . Consequently, one of the coefficients in (1) must be zero ( $\gamma_{z,xy} = -\gamma_{z,yx} = -\gamma_{x,xy} = 0$ ), and the other two are equal, but opposite in sign:  $\gamma_{x,yz} = -\gamma_{y,zx}$ . The same result is obtained for the class  $D_6$ .

The class  $S_4$  includes the transformations  $x \rightarrow y, y \rightarrow -x, z \rightarrow -z$  and  $x \rightarrow -x, y \rightarrow -y, z \rightarrow z$ . The non-zero components are  $\gamma_{z,xy}, \gamma_{x,yz} = \gamma_{y,zx}, \gamma_{z,xx} = -\gamma_{z,yy}, \gamma_{x,zx} = -\gamma_{y,zy}$ . One of these can be made to vanish by a suitable choice of the  $x$  and  $y$  axes.

The class  $D_3$  has one threefold axis of symmetry (the  $z$ -axis, say), and three twofold axes lying in the  $xy$ -plane; let one of these be the  $x$ -axis. To find the restrictions imposed by the presence of a threefold axis, we make a formal transformation by introducing the complex "coordinates"  $\xi = x + iy, \eta = x - iy$ ; the coordinate  $z$  remains unchanged. We must also transform the tensor  $\gamma_{i,kl}$  to these new coordinates, in which the suffixes take the values  $\xi, \eta, z$ . In a rotation through  $120^\circ$  about the  $z$ -axis these coordinates undergo the transformation  $\xi \rightarrow \xi e^{2\pi i/3}, \eta \rightarrow \eta e^{-2\pi i/3}, z \rightarrow z$ . The only components of the tensor  $\gamma_{i,kl}$  which remain unchanged and so may be different from zero are  $\gamma_{z,\xi\xi}, \gamma_{\eta,z\xi}, \gamma_{\xi,z\eta}, \gamma_{\xi,\xi\xi}, \gamma_{\eta,\eta\eta}$  and  $\gamma_{z,zz}$ . A rotation through  $180^\circ$  about the  $x$ -axis gives the transformation  $x \rightarrow x, y \rightarrow -y, z \rightarrow -z$ , or  $\xi \rightarrow \eta, \eta \rightarrow \xi, z \rightarrow -z$ ;  $\gamma_{z,\xi\xi}$  and  $\gamma_{z,zz}$  change sign and so must be zero, while the remaining components listed above are mutually transformed in pairs, giving  $\gamma_{\eta,z\xi} = -\gamma_{\xi,z\eta}, \gamma_{\xi,\xi\xi} = \gamma_{\eta,\eta\eta}$ . In order to write an expression of  $\tilde{\Phi}_{\text{pie}}$ , we must form the sum  $-\gamma_{i,kl} E_i \sigma_{kl}$ , in which the suffixes take the values  $\xi, \eta, z$ :

$$\tilde{\Phi}_{\text{pie}} = -2\gamma_{\eta,z\xi}(E_\eta \sigma_{z\xi} - E_\xi \sigma_{z\eta}) - \gamma_{\xi,\xi\xi}(E_\xi \sigma_{\xi\xi} + E_\eta \sigma_{\eta\eta}).$$

Here the components  $E_i$  and  $\sigma_{ik}$  in the coordinates  $\xi, \eta, z$  must also be expressed in terms of those in the original coordinates  $x, y, z$ . This is easily done by using the fact that the components of a tensor are transformed as the products of the corresponding coordinates. Hence, for example, from  $\xi\xi = xx - yy + 2ixy$ , we have  $\sigma_{\xi\xi} = \sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy}$ . The result is

$$\tilde{\Phi}_{\text{pie}} = 2a(E_y \sigma_{zx} - E_x \sigma_{zy}) + b[2E_y \sigma_{xy} - E_x(\sigma_{xx} - \sigma_{yy})], \quad (2)$$

where  $a = 2i\gamma_{\eta,z\xi}$  and  $b = 2\gamma_{\xi,\xi\xi}$  are real constants. The relations between the components  $\gamma_{i,kl}$  in the coordinates  $x, y, z$  are, as we see from (2),‡

$$\gamma_{y,zx} = -\gamma_{x,zy} \equiv -a, \quad \gamma_{y,xy} = -\gamma_{x,xx} = \gamma_{x,yy} \equiv -b.$$

The class  $D_{3h}$  is obtained from the class  $D_3$  by adding a plane of symmetry (the  $xy$ -plane) perpendicular to the threefold axis. Reflection in this plane changes the sign of  $z$ , and so  $\gamma_{\eta,z\xi} = 0$ , so that only the term with the coefficient  $b$  remains in (2).

The class  $C_{3h}$  has a threefold axis and a plane of symmetry perpendicular to it. Reflection in this plane changes the sign of  $z$ , and so all components  $\gamma_{i,kl}$  whose suffixes contain  $z$  an odd number of times must be zero. Taking into account also the restrictions derived above which are imposed by the threefold axis of symmetry, we find that only the two components  $\gamma_{\eta,\eta\eta}$  and  $\gamma_{\xi,\xi\xi}$  are not zero. These quantities must be complex conjugates in order that  $\tilde{\Phi}$

† To avoid misunderstanding it should be recalled that, if we calculate the components of the strain tensor  $u_{ik}$  by direct differentiation of the actual expression for  $\tilde{\Phi}$  with respect to  $\sigma_{ik}$ , the derivatives with respect to components  $\sigma_{ik}$  with  $i \neq k$  give twice the corresponding components  $u_{ik}$ , because the expressions  $u_{ik} = -\partial\tilde{\Phi}/\partial\sigma_{ik}$  are essentially meaningful only as representing the fact that  $d\tilde{\Phi} = -u_{ik}d\sigma_{ik}$ , and the terms containing the differentials of non-diagonal components of the symmetrical tensor  $\sigma_{ik}$  appear twice in the sum  $u_{ik}d\sigma_{ik}$ .

‡ In non-orthogonal coordinates such as  $\xi, \eta, z$  the covariant and contravariant components of tensors must be distinguished. This should have been done in returning to the original coordinates  $x, y, z$ : if the components  $E_i$  and  $\sigma_{kl}$  transform contravariantly, then those of the tensor  $\gamma_{i,kl}$  transform covariantly. We avoid this necessity, however, by obtaining the required relations between the components  $\gamma_{i,kl}$  in the coordinates  $x, y, z$  directly from the form of the scalar combination (2).

should be real. Putting  $2\gamma_{\eta,\eta\eta} = a - ib$ ,  $2\gamma_{\xi,\xi\xi} = a + ib$ , we find

$$\Phi_{\text{pie}} = a[2E_y\sigma_{xy} - E_x(\sigma_{xx} - \sigma_{yy})] + b[2E_x\sigma_{xy} + E_y(\sigma_{xx} - \sigma_{yy})]. \quad (3)$$

Either  $a$  or  $b$  can be made to vanish by a suitable choice of the  $x$  and  $y$  axes.

**PROBLEM 2.** The same as Problem 1, but for the crystal classes which allow pyroelectricity.

**SOLUTION.** Let the  $z$ -axis be the twofold, threefold, fourfold or sixfold axis of symmetry, or in the class  $C_s$  be perpendicular to the plane of symmetry. In the classes  $C_{nv}$  the  $xz$ -plane is a plane of symmetry.

We give below for each class all the components  $\gamma_{i,kl}$  which are not zero.

Class  $C_1$ : all  $\gamma_{i,kl}$ .

$C_s$ : all those in which the suffix  $z$  appears twice or not at all.

$C_{2v}$ :  $\gamma_{z,xx}, \gamma_{z,yy}, \gamma_{z,zz}, \gamma_{x,xz}, \gamma_{y,yz}$ .

$C_2$ : the same, together with  $\gamma_{x,yz}, \gamma_{y,xz}, \gamma_{z,xy}$ .

$C_{4v}$ :  $\gamma_{z,xx} = \gamma_{z,yy}, \gamma_{z,zz}, \gamma_{x,xz} = \gamma_{y,yz}$ .

$C_4$ : the same, together with  $\gamma_{x,yz} = -\gamma_{y,xz}$ .

$C_{3v}$ :  $\gamma_{z,zz}, \gamma_{x,xz} = \gamma_{y,yz}, \gamma_{x,xx} = -\gamma_{x,yy} = -\gamma_{y,xy}, \gamma_{z,xx} = \gamma_{z,yy}$ .

$C_3$ : the same, together with  $\gamma_{x,yz} = -\gamma_{y,xz}, \gamma_{y,xx} = -\gamma_{y,yy} = \gamma_{x,xy}$ .

$C_{6v}$ :  $\gamma_{z,zz}, \gamma_{x,xz} = \gamma_{y,yz}, \gamma_{z,xx} = \gamma_{z,yy}$ .

$C_6$ : the same, together with  $\gamma_{x,yz} = -\gamma_{y,xz}$ .

By a suitable choice of directions of the  $x, y, z$  axes three more components can be made zero in the class  $C_1$ , and by a choice of the  $x$  and  $y$  axes one more component can be made zero in the classes  $C_s, C_2, C_3$ ; in  $C_4$  and  $C_6$ , the expression  $\gamma_{i,kl}E_i\sigma_{kl}$  is invariant under rotation through any angle about the  $z$  axis, and therefore no further reduction in the number of non-zero components  $\gamma_{i,kl}$  is possible.

**PROBLEM 3.** Determine Young's modulus (the coefficient of proportionality between the extending stress and the relative extension) for a flat slab of a non-pyroelectric piezoelectric in the following cases: (a) where the slab is stretched by the plates of a short-circuited capacitor, (b) where it is stretched by those of an uncharged capacitor, (c) where it is stretched parallel to its plane with no external field.

**SOLUTION.** (a) In this case the field  $\mathbf{E}$  inside the slab is zero. The only non-zero component of the tensor  $\sigma_{ik}$  is the extending stress  $\sigma_{zz}$  (the  $z$ -axis being perpendicular to the slab).† From (17.8) we have  $u_{zz} = \mu_{zzzz}\sigma_{zz}$ , whence Young's modulus is  $E = 1/\mu_{zzzz}$ .

(b) In this case we have in the slab  $E_x = E_y = 0, D_z = 0$ . From (17.6) and (17.8) we have  $D_z = \epsilon_{zz}E_z + 4\pi\gamma_{z,zz}\sigma_{zz} = 0, u_{zz} = \mu_{zzzz}\sigma_{zz} + \gamma_{z,zz}E_z$ . Eliminating  $E_z$ , we obtain  $1/E = \mu_{zzzz} - 4\pi\gamma_{z,zz}^2/\epsilon_{zz}$ .

(c) In this case also,  $E_x = E_y = 0, D_z = 0$ , but the extension is along the  $x$ -axis, say. Here we have  $D_z = \epsilon_{zz}E_z + 4\pi\gamma_{z,xx}\sigma_{xx} = 0, u_{xx} = \mu_{xxxx}\sigma_{xx} + \gamma_{z,xx}E_z$ . Eliminating  $E_z$ , we obtain  $1/E = \mu_{xxxx} - 4\pi\gamma_{z,xx}^2/\epsilon_{zz}$ .

**PROBLEM 4.** Obtain an equation for the velocity of sound in a piezoelectric medium.

**SOLUTION.** In this problem it is more convenient to use  $u_{ik}$  as the independent variables, instead of  $\sigma_{ik}$ . We write  $\tilde{F}$  in the form

$$\tilde{F} = F_0 + \frac{1}{2}\lambda_{iklm}u_{ik}u_{lm} - \frac{1}{8\pi}\epsilon_{ik}E_iE_k - \frac{1}{4\pi}E_iD_{0i} + \beta_{i,kl}E_iu_{kl},$$

whence

$$\sigma_{ik} = \partial\tilde{F}/\partial u_{ik} = \lambda_{iklm}u_{lm} + \beta_{l,ik}E_l.$$

The equations of motion from the theory of elasticity are

$$\rho\ddot{u}_i = \frac{\partial\sigma_{ik}}{\partial x_k} = \lambda_{iklm}\frac{\partial u_{lm}}{\partial x_k} + \beta_{l,ik}\frac{\partial E_l}{\partial x_k}, \quad (4)$$

where  $\rho$  is the density of the medium, and  $\mathbf{u}$  is the displacement vector, related to  $u_{ik}$  by

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right).$$

The equation  $\text{div } \mathbf{D} = 0$  gives

$$\epsilon_{ik}\frac{\partial E_k}{\partial x_i} - 4\pi\beta_{i,kl}\frac{\partial u_{kl}}{\partial x_i} = 0, \quad (5)$$

† It is not assumed to coincide with any particular crystallographic direction.

and the field can be expressed in terms of the field potential:  $E_i = -\partial\phi/\partial x_i$ , which takes into account the equation  $\text{curl } \mathbf{E} = 0$ .

In a plane sound wave,  $u$  and  $\phi$  are proportional to  $\exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$ , and we find from the above equations that

$$\begin{aligned}\rho\omega^2 u_i &= \lambda_{iklm} k_k k_l u_m - \beta_{i,ik} k_k k_l \phi, \\ \epsilon_{ik} k_i k_k \phi + 4\pi\beta_{i,ki} k_i k_k u_i &= 0.\end{aligned}$$

Eliminating  $\phi$ , we can write the condition of compatibility of the resulting equations for  $u_i$  as

$$\det|\rho\omega^2 \delta_{ik} - \lambda_{iklm} k_l k_m - 4\pi(\beta_{i,mi} k_l k_m)(\beta_{p,qk} k_p k_q)/\epsilon_{rs} k_r k_s| = 0.$$

For any given direction of the wave vector  $\mathbf{k}$ , this equation determines three phase velocities of sound  $\omega/k$ , which are in general different. A characteristic property of a piezoelectric medium is the involved relation between the velocity and direction of the wave.

**PROBLEM 5.** A piezoelectric crystal of the class  $C_{6v}$  has a plane boundary (the  $xz$ -plane) which passes through the axis of symmetry (the  $z$ -axis). Find the speed of surface waves propagated at right angles to the symmetry axis (in the  $x$ -direction), in which there are oscillations of the displacement  $u_z$  and the electric field potential  $\phi$  (J. L. Bleustein, 1968; Yu. V. Gulyaev, 1969).

**SOLUTION.** Under the conditions considered, two equations involving only  $u_z$  and  $\phi$  separate out from (4) and (5); these quantities depend on the coordinates  $x$  and  $y$ , and on the time  $t$ , but not on  $z$ . The non-zero components of the stress tensor and the induction vector are

$$\begin{aligned}\sigma_{zx} &= \beta E_x + 2\lambda u_{zx}, & \sigma_{zy} &= \beta E_y + 2\lambda u_{zy}, \\ D_x &= -8\pi\beta u_{zx} + \epsilon E_x, & D_y &= -8\pi\beta u_{zy} + \epsilon E_y,\end{aligned}$$

with

$$\begin{aligned}u_{zx} &= \frac{1}{2}\partial u_z/\partial x, & u_{zy} &= \frac{1}{2}\partial u_z/\partial y, \\ E_x &= -\partial\phi/\partial x, & E_y &= -\partial\phi/\partial y,\end{aligned}$$

and writing for brevity  $\beta_{x,xz} = \beta_{y,yz} \equiv \beta$ ,  $\lambda_{xxzx} = \lambda_{yyzy} \equiv \lambda$ ,  $\epsilon_{xx} = \epsilon_{yy} \equiv \epsilon$ ; the constant pyroelectric induction  $D_z = D_0$  does not appear in the equations or the boundary conditions.

Equation (5) and the  $z$ -component of equation (4) give, in the region occupied by the piezoelectric medium (the half-space  $y > 0$ ),

$$4\pi\beta \Delta u_z + \epsilon \Delta \phi^{(i)} = 0, \quad \rho \ddot{u}_z = -\beta \Delta \phi^{(i)} + \lambda \Delta u_z,$$

where  $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ ; these may be rewritten as

$$\rho \ddot{u}_z = \bar{\lambda} \Delta u_z, \quad \Delta \psi = 0, \quad (6)$$

where

$$\bar{\lambda} = \lambda + 4\pi\beta^2/\epsilon, \quad \psi = (4\pi\beta/\epsilon)u_z + \phi^{(i)}.$$

In the vacuum (the half-space  $y < 0$ ), the potential  $\phi^{(e)}$  satisfies the equation

$$\Delta \phi^{(e)} = 0. \quad (7)$$

These equations are to be solved with the following boundary conditions: at the surface of the medium,

$$\phi^{(i)} = \phi^{(e)}, \quad \sigma_{zy} = 0, \quad D_y^{(i)} = -\partial\phi^{(e)}/\partial y \quad \text{for } y = 0. \quad (8)$$

and far from the surface

$$u_z \rightarrow 0 \quad \text{as } y \rightarrow \infty; \quad \phi \rightarrow 0 \quad \text{as } y \rightarrow \pm \infty.$$

We seek the solution in the form

$$u_z = A e^{-\kappa y} e^{i(kx - \omega t)}, \quad \psi = B e^{-\kappa y} e^{i(kx - \omega t)}, \quad \phi^{(e)} = C e^{\kappa y} e^{i(kx - \omega t)},$$

with

$$\rho\omega^2 = \bar{\lambda}(k^2 - \kappa^2). \quad (9)$$

Equations (6) and (7) and the conditions at infinity are then satisfied, and the conditions (8) give three linear homogeneous equations for  $A$ ,  $B$  and  $C$ ; the condition for these to have a solution is

$$\kappa = 4\pi\beta^2 k / \bar{\lambda}\epsilon(1 + \epsilon) \equiv \Lambda k.$$

Finally, substitution in (9) gives the phase velocity of the waves:

$$\omega/k = [(\bar{\lambda}/\rho)(1 - \Lambda^2)]^{\frac{1}{2}}.$$

The surface propagation of these waves is restricted to piezoelectric media. As  $\beta \rightarrow 0$ , the penetration depth  $1/\kappa \rightarrow \infty$ , and a bulk wave is formed.

### §18. Thermodynamic inequalities

According to the formulae of §10, the total free energy can be written as the integral

$$\mathcal{F} = \int F(T, \rho, \mathbf{D}) dV, \quad (18.1)$$

taken over all space. We shall suppose that the function  $\mathbf{D}(\mathbf{r})$  which appears in the integrand satisfies only the equation

$$\text{div } \mathbf{D} = 0 \quad (18.2)$$

inside a dielectric and the condition

$$\oint \mathbf{D} \cdot d\mathbf{f} = 4\pi e \quad (18.3)$$

on the surface of a conductor which carries a given charge. These equations establish the relation between the field and its sources. Otherwise we regard the function  $\mathbf{D}(\mathbf{r})$  as arbitrary, and in particular we do not require it to satisfy the second field equation  $\text{curl } \mathbf{E} = 0$  (where  $\mathbf{E} = 4\pi \partial F / \partial \mathbf{D}$ ) or the boundary condition  $\phi = \text{constant}$  on the surface of a conductor. We shall show that these equations can then be obtained from the condition that the integral (18.1) be a minimum with respect to changes in the function  $\mathbf{D}(\mathbf{r})$  which satisfy equations (18.2) and (18.3). It should be emphasized that the possibility of this derivation is not *a priori* evident, since the field distributions which come into consideration in determining the minimum of the integral (18.1) do not necessarily correspond to physically possible states (because they do not satisfy all the field equations), whereas, in the thermodynamic condition that the free energy be a minimum, only the various physically possible states are considered.

The problem of finding the minimum of the integral (18.1) with the subsidiary conditions (18.2) and (18.3) is solved by Lagrange's method of multipliers. We multiply the variation of the condition (18.2) by some as yet undetermined function  $-\phi/4\pi$  of the coordinates, and that of the condition (18.3) by some undetermined constant  $\phi_0/4\pi$ , and then equate to zero the sum of variations

$$\int \delta F dV - \frac{1}{4\pi} \int \phi \text{div } \delta \mathbf{D} dV + \frac{\phi_0}{4\pi} \oint \delta \mathbf{D} \cdot d\mathbf{f} = 0.$$

In the first term we write†

$$\delta F = (\partial F / \partial \mathbf{D})_{T, \rho} \cdot \delta \mathbf{D} = \mathbf{E} \cdot \delta \mathbf{D} / 4\pi,$$

and the second can be integrated by parts:  $\int \phi \text{div } \delta \mathbf{D} dV = \oint \phi \delta \mathbf{D} \cdot d\mathbf{f} - \int \delta \mathbf{D} \cdot \text{grad } \phi dV$ .

The result is

$$\int (\mathbf{E} + \text{grad } \phi) \cdot \delta \mathbf{D} dV + \oint (\phi_0 - \phi) \delta \mathbf{D} \cdot d\mathbf{f} = 0.$$

† The free energy is the minimum for a given temperature. The variation is with respect to two independent quantities  $\mathbf{D}$  and  $\rho$ . Here we are interested only in the result of varying with respect to  $\mathbf{D}$ . The variation of the integral (18.1) with respect to density (with the subsidiary condition of constant total mass of the body) gives one of the usual conditions of thermal equilibrium, namely the constancy of the chemical potential  $\zeta$ .

Hence we conclude that, throughout the volume, we must have  $\mathbf{E} = -\text{grad } \phi$  (and so  $\text{curl } \mathbf{E} = 0$ ), and on the surface of a conductor  $\phi = \phi_0 = \text{constant}$ . These are the correct equations for the field, and the Lagrange multiplier  $\phi$  is its potential.

Similarly it can be shown that the equations for the electric induction are obtained from the condition that the integral  $\tilde{\mathcal{F}} = \int \tilde{F}(T, \rho, \mathbf{E}) dV$  be a maximum, in which the function  $\mathbf{E}(\mathbf{r})$  is varied with the subsidiary conditions  $\mathbf{E} = -\text{grad } \phi$  and  $\phi = \text{constant}$  on the surface of a conductor.† For

$$\begin{aligned}\delta \tilde{\mathcal{F}} &= \int (\partial \tilde{F} / \partial \mathbf{E}) \cdot \delta \mathbf{E} dV = \int \mathbf{D} \cdot \text{grad } \delta \phi dV / 4\pi \\ &= \oint \delta \phi \mathbf{D} \cdot d\mathbf{f} / 4\pi - \int \delta \phi \text{div } \mathbf{D} dV / 4\pi = 0.\end{aligned}$$

The first integral is zero because  $\delta \phi = 0$  on the surface, and from the second we find the required equation  $\text{div } \mathbf{D} = 0$ , since  $\delta \phi$  is arbitrary in the volume.

If the body is not in an external electric field (in particular, if there are no charged conductors), it may be possible to formulate the condition of thermodynamic equilibrium as the condition that the total free energy (18.1) have an absolute (unconditional) minimum. This amounts to the condition that the free energy density  $F$  be a minimum as a function of the independent variable  $\mathbf{D}$ :  $\partial F / \partial \mathbf{D} = \mathbf{E} / 4\pi = 0$ , i.e. the field must be zero in all space. If it is possible to find a distribution of the induction such that  $\text{div } \mathbf{D} = 0$ , this state will correspond to thermodynamic equilibrium.‡

Equating to zero the first variation of the free energy, we find necessary but not sufficient conditions for this energy to be a minimum. The determination of the sufficient conditions requires a discussion of the second variation. These conditions take the form of certain inequalities (called *thermodynamic inequalities*) and are the conditions which ensure the stability of the state of the body (see SP 1, §21).

When  $\mathbf{D} = \epsilon \mathbf{E}$ , the situation is much simplified, and the thermodynamic inequality of interest here (relating to the dielectric properties of the body) becomes evident. The total free energy is  $\mathcal{F}_0 + \int (D^2 / 8\pi\epsilon) dV$ . It is clear that this can have a minimum only if  $\epsilon > 0$ , since otherwise the integral could be made to take any large negative value by making  $D^2$  large enough. Thus in this case nothing new is learnt, since we know already that the permittivity must in fact be not only positive but greater than unity (see §14).

In the general case of an arbitrary relation between  $\mathbf{D}$  and  $\mathbf{E}$ , however, it is necessary to consider the second variation of the integral (18.1), and to vary simultaneously both  $\mathbf{D}$  and  $\rho$  (leaving only the temperature constant). In an isotropic body,  $F(T, \rho, \mathbf{D})$  depends only on the magnitude of the vector  $\mathbf{D}$ , but its three components vary independently. We take the direction of the vector  $\mathbf{D}$  before variation as the  $z$ -axis. Then the change in the magnitude of  $\mathbf{D}$  is given in terms of the changes in its components, as far as the second-order terms, by  $\delta D = \delta D_z + (\delta D_x)^2 / 2D + (\delta D_y)^2 / 2D$ . The first and second variations of the integral (18.1)

† The point that the thermodynamic potential  $\tilde{\mathcal{F}}$  has a maximum, not a minimum like  $\mathcal{F}$ , with respect to the variable  $\mathbf{E}$  or  $\mathbf{D}$  is a general one, and is accounted for as follows. Let the equilibrium value of a variable  $x$ , say  $x = 0$ , be determined by the condition of thermodynamic equilibrium. Then the free energy  $\mathcal{F}$  has, for a given  $T$  and  $V$ , a minimum at  $x = 0$ . We thus have, at  $x = 0$ ,  $X \equiv (\partial \mathcal{F} / \partial x)_{V,T} = 0$ , and near that point  $X = \alpha x$ ,  $\mathcal{F} = \mathcal{F}_0 + \frac{1}{2} \alpha x^2$  with  $\alpha > 0$ . With the thermodynamic potential  $\tilde{\mathcal{F}} = \mathcal{F} - xX$ , this gives  $\tilde{\mathcal{F}} = \mathcal{F}_0 - \frac{1}{2} \alpha x^2 = \mathcal{F}_0 - X^2 / 2\alpha$ , so that in equilibrium  $\tilde{\mathcal{F}}$  has a maximum with respect to  $x$  or  $X$ . But both  $\mathcal{F}$  and  $\tilde{\mathcal{F}}$  have minima with respect to any other variables  $y$  that are independent of  $x$ .

‡ Here we are considering bodies in which  $\mathbf{D}$  need not be zero even if  $\mathbf{E} = 0$  (see §19). Otherwise we have simply the trivial result  $\mathbf{E} = \mathbf{D} = 0$  in all space.

are both contained in the expression

$$\int \left\{ \frac{\partial F}{\partial D} \delta D + \frac{\partial F}{\partial \rho} \delta \rho + \frac{1}{2} \frac{\partial^2 F}{\partial D^2} (\delta D)^2 + \frac{\partial^2 F}{\partial D \partial \rho} \delta D \delta \rho + \frac{1}{2} \frac{\partial^2 F}{\partial \rho^2} (\delta \rho)^2 \right\} dV.$$

Substituting  $\delta D$  and collecting the second-order terms, we find the second variation

$$\int \frac{1}{2D} \frac{\partial F}{\partial D} [(\delta D_x)^2 + (\delta D_y)^2] dV + \int \left\{ \frac{1}{2} \frac{\partial^2 F}{\partial D^2} (\delta D_z)^2 + \frac{\partial^2 F}{\partial D \partial \rho} \delta D_z \delta \rho + \frac{1}{2} \frac{\partial^2 F}{\partial \rho^2} (\delta \rho)^2 \right\} dV. \quad (18.4)$$

These two terms are independent. The first is positive if  $(1/D)\partial F/\partial D > 0$ . But  $\partial F/\partial \mathbf{D} = \mathbf{E}/4\pi$ , so that the derivative  $\partial F/\partial D$  is positive or negative according as the vectors  $\mathbf{D}$  and  $\mathbf{E}$  are in the same or opposite directions. Thus these vectors must be in the same direction.

The conditions for the second term in (18.4) to be positive are

$$\partial^2 F / \partial \rho^2 > 0, \quad (18.5)$$

$$\frac{\partial^2 F}{\partial \rho^2} \frac{\partial^2 F}{\partial D^2} - \left( \frac{\partial^2 F}{\partial \rho \partial D} \right)^2 > 0. \quad (18.6)$$

Since  $\partial F/\partial \rho = \zeta$ ,  $\partial F/\partial D = E/4\pi$ , the first of these gives

$$(\partial \zeta / \partial \rho)_{D,T} > 0, \quad (18.7)$$

and the second can be rewritten as a Jacobian:

$$\frac{\partial(\partial F/\partial D, \partial F/\partial \rho)}{\partial(D, \rho)} = \frac{1}{4\pi} \frac{\partial(E, \zeta)}{\partial(D, \rho)} > 0.$$

Changing from the variables  $D, \rho$  to  $D, \zeta$ , we have

$$\frac{\partial(E, \zeta)}{\partial(D, \rho)} = \frac{\partial(E, \zeta)}{\partial(D, \zeta)} \frac{\partial(D, \zeta)}{\partial(D, \rho)} = \left( \frac{\partial E}{\partial D} \right)_\zeta \left( \frac{\partial \zeta}{\partial \rho} \right)_D > 0;$$

by (18.7), this gives

$$(\partial E / \partial D)_{\zeta,T} > 0. \quad (18.8)$$

Thus we have derived the required thermodynamic inequalities. In the absence of a field, the inequality (18.7) becomes the usual condition that the isothermal compressibility is positive:  $(\partial P / \partial \rho)_T > 0$ .† The inequality (18.8) gives  $\epsilon > 0$ , since when  $E \rightarrow 0$  the induction  $D \rightarrow \epsilon E$ .

Of the two inequalities (18.5), (18.6) the latter is the stronger; it may be violated while the

† It should be recalled that, in the absence of a field,  $\zeta$  is the thermodynamic potential of unit mass and, by the ordinary thermodynamic relations, its differential

$$d\zeta = dP/\rho - (S/\rho)dT,$$

so that  $(\partial \zeta / \partial \rho)_T = (1/\rho)(\partial P / \partial \rho)_T$ . In the above derivation the second of the ordinary thermodynamic inequalities (that the specific heat is positive) is ignored.

first is not, whereas the reverse is impossible. The equation

$$\frac{\partial^2 F}{\partial \rho^2} \frac{\partial^2 F}{\partial D^2} - \left( \frac{\partial^2 F}{\partial \rho \partial D} \right)^2 = \frac{1}{4\pi} \frac{\partial(E, \zeta)}{\partial(D, \rho)} = 0$$

corresponds to what is called the *critical state* (see SP 1, §83). This condition is more conveniently written in a different form by multiplying it by the non-zero factor  $\partial(D, \rho)/\partial(E, \rho)$ :

$$\partial(E, \zeta)/\partial(E, \rho) = (\partial\zeta/\partial\rho)_{E,T} = 0. \quad (18.9)$$

The critical states occupy a curve in the  $ET$ -plane, which is a singularity of the thermodynamic functions of the body, just as the critical point is a singularity in the absence of the field.

## §19. Ferroelectrics

The various crystalline modifications of a given substance may include some which are pyroelectric and some which are not. If the change from one to the other takes place by means of a second-order phase transition, then near the transition point the substance has a number of unusual properties which distinguish it from ordinary pyroelectrics; these are called *ferroelectric* substances.

In an ordinary pyroelectric crystal, a change in the direction of the spontaneous polarization involves a considerable reconstruction of the crystal lattice. Even if the final result of this reconstruction is energetically favourable, its realization may still be impossible because it would require the surmounting of very high energy barriers.

In a ferroelectric body, however, the situation is quite different because, near a second-order phase transition point, the arrangement of the atoms in the crystal lattice of the pyroelectric phase is only slightly different from the arrangement in the non-pyroelectric lattice (and so the spontaneous polarization also is small). For this reason the change in direction of the spontaneous polarization here requires only a slight reconstruction of the lattice and can occur quite easily.

The actual nature of the ferroelectric properties of a body depends on its crystal symmetry. The direction of the spontaneous polarization of the pyroelectric phase (which we shall call the *ferroelectric axis*) is determined by the structure of the non-pyroelectric phase beyond the transition point. In some cases it is uniquely determined, in the sense that the ferroelectric axis can lie in only one, crystallographically determinate, direction; the direction of the spontaneous polarization is then determined apart from sign, since in the non-pyroelectric phase the two opposite directions parallel to the ferroelectric axis must be equivalent (otherwise this form of the crystal would also be pyroelectric). In other cases, the symmetry of the non-pyroelectric phase may be such as to allow spontaneous polarization in any of several crystallographically equivalent directions.†

† An instance of the first type is sodium potassium tartrate, whose non-pyroelectric phase has orthorhombic symmetry. The ferroelectric axis appears in it in a completely definite crystallographic direction (one of the twofold axes), and the lattice becomes monoclinic.

An instance of the second type is barium titanate. Its non-pyroelectric modification has a cubic lattice, and any of the three cubic axes may become the ferroelectric axis. After the spontaneous polarization has appeared at the transition point, these three directions, of course, are no longer equivalent. The ferroelectric axis becomes the only fourfold axis, and the lattice becomes tetragonal.



The occurrence of polarization is always associated with a reduction in the symmetry of the crystal. We can therefore refer to pyroelectric and non-pyroelectric phases as the unsymmetrical and symmetrical phases respectively, using the terminology of SP 1, §142.

We shall show how the theory of ferroelectricity can be developed in terms of the general theory of Landau second-order phase transitions, as was first done by V. L. Ginzburg (1945).†

We take the dielectric polarization vector  $\mathbf{P}$  of the substance as the order parameter, whose magnitude determines the difference between the unsymmetrical and symmetrical phase lattice structures. This means that  $\mathbf{P}$  will be regarded as an independent thermodynamic variable whose actual value (as a function of the temperature, the field, etc.) is then determined from the condition of thermal equilibrium, namely that the thermodynamic potential be a minimum.

Let us consider first the case where the position of the ferroelectric axis, which we take as the  $z$ -axis, is uniquely determined. The dielectric properties of the crystal in the  $x$  and  $y$  directions then exhibit no anomalies, and to investigate the properties in the  $z$ -direction we need consider only those terms in the thermodynamic potential which contain  $P_z$ . Near the transition point, the order parameter  $P_z$  is small and the thermodynamic potential  $\Phi$  can be expanded in powers of  $P_z$ . Since the two directions of the  $z$ -axis are equivalent, the expansion cannot depend on the sign of  $P_z$ , and therefore contains only even powers. As far as the fourth-order terms, we have

$$\Phi = \Phi_0 + AP_z^2 + BP_z^4. \quad (19.1)$$

In the symmetrical phase,  $A > 0$ , and  $P_z = 0$  corresponds to a minimum of the thermodynamic potential. For spontaneous polarization to occur,  $A$  must be negative; it is therefore zero at the phase transition point. The Landau theory assumes that  $A(T)$  can be expanded in integral powers of  $T - T_c$ , where  $T_c$  is the phase transition temperature; near this point, we write  $A = a(T - T_c)$ ,  $a$  being a constant (independent of the temperature). We shall take the specific case where  $a > 0$ , so that the unsymmetrical phase corresponds to temperatures  $T < T_c$ . The condition for the state to be stable at the point  $T = T_c$  itself is that the coefficient  $B$  be positive at that point and therefore throughout a neighbourhood of it. In what follows,  $B$  will denote  $B(T_c)$ .

If the electric field in the body is not zero, further terms appear in the thermodynamic potential. To find these, we start from the relation

$$4\pi\partial\tilde{\Phi}/\partial\mathbf{E} = -\mathbf{D} = -\mathbf{E} - 4\pi\mathbf{P}. \quad (19.2)$$

Integration with a fixed value of the independent variable  $\mathbf{P}$ , using the fact that  $\Phi$  and  $\tilde{\Phi}$  are the same when  $\mathbf{E} = 0$ , gives

$$\tilde{\Phi}(\mathbf{P}, \mathbf{E}) = \Phi(\mathbf{P}, 0) - \mathbf{E} \cdot \mathbf{P} - E^2/8\pi.$$

With the electric field in the  $z$ -direction, and  $\Phi(\mathbf{P}, 0)$  from (19.1), we have

$$\tilde{\Phi} = \Phi_0 + a(T - T_c)P_z^2 + BP_z^4 - E_z P_z - E_z^2/8\pi. \quad (19.3)$$

† The Landau theory certainly becomes invalid in a neighbourhood of the transition point. The question of when this happens in ferroelectrics needs a specific analysis of experimental results, and lies outside the scope of the present book. Actually, many ferroelectric transitions are not second-order, but first-order ones close to being second-order. This seems to be due to the fluctuation effect mentioned in SP 1, end of §146.



The presence of the term  $-E_z P_z$  has the result that in any field  $E_z$ , however weak, the order parameter  $P_z$  becomes different from zero at every temperature; the field polarizes the non-pyroelectric phase and thus reduces its symmetry. The qualitative difference between the two phases thereby disappears, and accordingly so does the discrete transition point; the transition is "smoothed out".†

The thermodynamic potential  $\tilde{\Phi}$  in equilibrium must be a minimum, for any given field  $E$ . Differentiation of (19.3) at constant  $E_z$  gives

$$2P_z a(T - T_c) + 4BP_z^3 = E_z. \quad (19.4)$$

This is the basic relation between the field and the polarization in a ferroelectric.‡

When  $T > T_c$ , in the non-pyroelectric phase,  $P_z$  vanishes with  $E_z$ . As  $E_z$  increases, the polarization at first increases linearly,  $P_z = \kappa E_z$ , with susceptibility

$$\kappa = 1/2a(T - T_c), \quad T > T_c, \quad (19.5)$$

which increases without limit as  $T \rightarrow T_c$ . The induction  $D_z = (1 + 4\pi\kappa)E_z$  also increases linearly with  $P_z$ . Near the transition point,  $\kappa$  is large, and we have to the same accuracy

$$\varepsilon \cong 4\pi\kappa = 2\pi/a(T - T_c). \quad (19.6)$$

In sufficiently strong fields, the polarization increases according to  $P_z = (E_z/4B)^{1/3}$ .

When  $T < T_c$ , in the pyroelectric phase,  $P_z = 0$  cannot correspond to a stable state. For  $E_z = 0$ , we find from (19.4) the spontaneous polarization of this phase,

$$P_{z0} = \pm \sqrt{[a(T_c - T)/2B]}. \quad (19.7)$$

The dielectric susceptibility of the phase can be found as the derivative  $dP_z/dE_z$  as  $E_z \rightarrow 0$ . From (19.4),

$$[-2(T_c - T)a + 12BP_z^2]dP_z/dE_z = 1, \quad (19.8)$$

and substitution of (19.7) gives

$$\kappa = [dP_z/dE_z]_{E_z=0} = 1/4a(T_c - T), \quad T < T_c. \quad (19.9)$$

This is half the susceptibility of the non-pyroelectric phase for the same value of  $|T_c - T|$ . In sufficiently weak fields, the polarization is  $P_z = P_{z0} + \kappa E_z$ , the induction is  $D_z = D_{z0} + \varepsilon E_z$ , where  $D_{z0} = 4P_{z0}$ , and the permittivity is

$$\varepsilon \cong 4\pi\kappa = \pi/a(T_c - T). \quad (19.10)$$

Figure 14 (p. 80) shows the function  $P_z(E_z)$  given by (19.4) for  $T < T_c$ . First of all, it should be noted that the dashed part  $cc'$  does not correspond to stable states: from (19.8) written in the form

$$(dP_z/dE_z)(\partial^2 \tilde{\Phi}/\partial P_z^2)_{E_z} = 1$$

we see that  $dP_z/dE_z < 0$  implies that  $\partial^2 \tilde{\Phi}/\partial P_z^2 < 0$ , i.e., the thermodynamic potential  $\tilde{\Phi}$  has a maximum, not a minimum. The ordinates of  $c$  and  $c'$  are given by the equation  $dE_z/dP_z = 0$ , and we conclude that the possible values of  $|P_z|$  in the pyroelectric phase are

† Cf. SP 1, §144. The discussion below is largely a repetition of the one given there.

‡ Expressing  $\mathbf{P}(\mathbf{E})$  by means of (19.4) and substituting in (19.3), we find the potential  $\tilde{\Phi}(\mathbf{E})$  as a function of  $\mathbf{E}$  only. From the condition  $\partial \tilde{\Phi}(\mathbf{P}, \mathbf{E})/\partial \mathbf{P} = 0$ , the equation  $\mathbf{D} = -4\pi \partial \tilde{\Phi}/\partial \mathbf{E}$  is valid both for  $\tilde{\Phi}(\mathbf{E})$  and for  $\tilde{\Phi}(\mathbf{P}, \mathbf{E})$  (which is differentiated at constant  $\mathbf{P}$ ).

restricted by the condition

$$P_z^2 > (T_c - T)a/6B. \quad (19.11)$$

If we consider states of a ferroelectric with given values of  $E_z$ , there is still an ambiguity in the value of  $P_z$ , in the range of abscissae between  $c$  and  $c'$ , and the question arises of the physical significance of the two values. We shall assume the ferroelectric to be a homogeneous flat slab, with the ferroelectric axis normal to it, lying between the plates of a capacitor, which are maintained at given potentials, i.e. which set up a given uniform field  $E = E_z$ .

For given potentials on the conductors, the condition of stability requires that the thermodynamic potential  $\tilde{\Phi}$  be a minimum. In particular, for  $\mathbf{E} = 0$  there are two states in which  $P_z$  has opposite signs (the points  $a$  and  $a'$  in Fig. 14) but  $\tilde{\Phi} (= \Phi)$  is the same. These two states, therefore, are equally stable, i.e. they are two phases which can coexist in contact.

Hence it is clear that the portions  $ac$  and  $a'c'$  of the curve correspond to states which are metastable but not absolutely stable. It is easy to see directly that the values of  $\tilde{\Phi}$  on  $ac$  and on  $a'c'$  are in fact greater than its values on  $a'b'$  and  $ab$  for the same value of  $E_z$ . The ordinates of  $a$  and  $a'$  are given by formula (19.7). Thus the range of metastability is

$$(T_c - T)a/6B < P_z^2 < (T_c - T)a/2B. \quad (19.12)$$

The existence of these two phases with  $\mathbf{E} = 0$  is very important, since it means that a ferroelectric body can be divided into a number of separate regions or *domains* in which the polarization is in opposite directions. On the surfaces separating these domains, the normal component of  $\mathbf{D}$  and the tangential component of  $\mathbf{E}$  must be continuous. The latter condition is satisfied identically, because  $\mathbf{E} = 0$ . From the former condition it follows that the domain boundaries must be parallel to the  $z$ -axis.

The actual shapes and sizes of the domains are determined by the condition that the total thermodynamic potential of the body should be a minimum.†

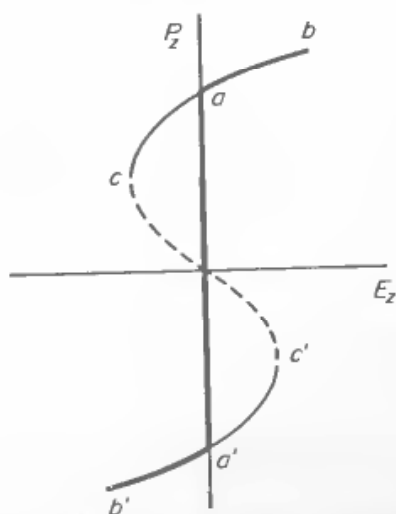


FIG. 14

† It should be emphasized that complete thermodynamic equilibrium is under consideration here. This can occur in ferroelectrics, but practically never does so in ordinary pyroelectrics because of the already mentioned difficulty of reorienting the polarization (and so of forming domains) in these materials. The shape and size of the domains will be discussed in §44 for the case of ferromagnets, which is in many respects analogous. We shall not pause to consider the specific features of the domain structure in ferroelectrics. They are due mainly to the rigid coupling of the direction of polarization to particular crystal axes, to the high dielectric susceptibility in comparison with the magnetic susceptibility of a ferromagnet, and to the greater role of striction effects.

If we are not interested in the details of the structure, and consider portions of the body which are large compared with the domains, we can use the polarization  $\bar{\mathbf{P}}$  averaged over such portions. Its component  $\bar{P}_z$  can evidently take values in the range between the ordinates of  $a$  and  $a'$  in Fig. 14, i.e.

$$-\sqrt{[(T_c - T)a/2B]} < \bar{P}_z < \sqrt{[(T_c - T)a/2B]}. \quad (19.13)$$

In other words, if  $P_z$  in Fig. 14 is taken as the polarization averaged in this way, the vertical segment  $aa'$  corresponds to the region of domain structure, and the thick curve  $baa'b'$  gives all stable states of the body.

Let us consider ferroelectrics which belong (in the non-pyroelectric phase) to the cubic system.<sup>†</sup> The cubic symmetry admits two independent fourth-order invariants formed from the components of the vector  $\mathbf{P}$ , which may be taken as  $(P_x^2 + P_y^2 + P_z^2)^2$  and  $P_x^2 P_y^2 + P_x^2 P_z^2 + P_y^2 P_z^2$ . Hence the expansion of the thermodynamic potential near the transition point (when  $\mathbf{E} = 0$ ) is of the form

$$\begin{aligned} \Phi = \Phi_0 + a(T - T_c)(P_x^2 + P_y^2 + P_z^2) + B(P_x^2 + P_y^2 + P_z^2)^2 \\ + C(P_x^2 P_y^2 + P_x^2 P_z^2 + P_y^2 P_z^2), \end{aligned} \quad (19.14)$$

where  $a$ ,  $B$ ,  $C$  are constants, and the  $x$ ,  $y$ ,  $z$  axes are along the three fourfold axes of symmetry.

The sum of the fourth-order terms in (19.14) must be essentially positive. Hence we must have

$$B > 0, \quad 3B + C > 0. \quad (19.15)$$

The spontaneous polarization of a ferroelectric when  $\mathbf{E} = 0$  is determined by the condition that  $\Phi$  should be a minimum as a function of  $\mathbf{P}$ . In particular, since the second-order term and the first of the fourth-order terms are independent of the direction of  $\mathbf{P}$ , the direction of the spontaneous polarization is determined by the condition that the next term be a minimum for a given absolute value  $P$ . Two cases are possible. If  $C > 0$ , the minimum of this term corresponds to  $\mathbf{P}$  being along any one of the axes  $x$ ,  $y$ ,  $z$ , i.e. along any of the three edges of the cube. If, however,  $C < 0$ , the minimum value occurs when  $\mathbf{P}$  is along any one of the spatial diagonals of the cube, i.e. when  $P_x^2 = P_y^2 = P_z^2 = \frac{1}{3}P^2$ . In the former case the pyroelectric phase of the ferroelectric has tetragonal symmetry, and in the latter case it has rhombohedral symmetry.

Let us consider in more detail, for example, the first case ( $C > 0$ ), and take as the  $z$ -axis the direction of the spontaneous polarization below the transition point. The magnitude  $P_0$  of this polarization is determined (when  $\mathbf{E} = 0$ ) by the minimum of the expression  $-a(T_c - T)P^2 + BP^4$ , whence

$$P_0^2 = a(T_c - T)/2B. \quad (19.16)$$

To find the polarization as a function of the field  $\mathbf{E}$ , we must add to (19.14) a term  $-\mathbf{P} \cdot \mathbf{E}$  (thus changing to the potential  $\tilde{\Phi}$ ) and equate to zero the derivative  $\partial \tilde{\Phi} / \partial \mathbf{P}$ .

When the field  $\mathbf{E}$  is weak,  $P_x$ ,  $P_y$ , and  $P_z - P_0$  are small. Omitting from the equations the

<sup>†</sup> The crystal classes  $T_h$  and  $O_h$  are envisaged here. The cubic classes  $T$  and  $T_d$  allow also a third-order invariant  $P_x P_y P_z$ ; under such conditions, the state with  $\mathbf{P} = 0$  certainly could not satisfy the stability condition (minimum  $\Phi$ ), and a second-order phase transition is therefore impossible. The symmetry of the class  $O$  (and that of  $T$ ) allows an invariant  $\mathbf{P} \cdot \text{curl } \mathbf{P}$  that is linear in the derivatives, and this gives rise to an incommensurate structure (cf. §52).

terms of the second and higher orders of smallness, and substituting  $P_{0z} = P_0$  from (19.16), we find the longitudinal polarization

$$P_z - P_0 = E_z/4a(T_c - T) \quad (19.17)$$

and the transverse polarization

$$P_x = BE_x/aC(T_c - T) \quad (19.18)$$

(and similarly for  $P_y$ ). Above the transition point, in the non-pyroelectric phase, the dielectric susceptibility of a cubic ferroelectric is the same in all directions:

$$\mathbf{P} = \mathbf{E}/2a(T - T_c). \quad (19.19)$$

Finally, let us briefly consider the elastic properties of ferroelectrics. According to its crystal class, the non-pyroelectric phase of a ferroelectric may or may not be piezoelectric.† Let us begin with the former case, and suppose that the symmetry admits a piezoelectric (linear) relation between the deformation and the polarization in the direction of the ferroelectric ( $z$ ) axis. These include the classes  $D_2$ ,  $D_{2d}$  and  $S_4$ ; in each case the polarization  $P_z$  appears in the piezoelectric part of the thermodynamic potential through a term  $-\gamma_{z,xy}P_z\sigma_{xy}$ .

In the elastic energy of these crystals, the component  $\sigma_{xy}$  appears in a term  $-\mu_{xyxy}\sigma_{xy}^2$ . Thus the thermodynamic potential near the transition point is

$$\tilde{\Phi} = \Phi_0 + a(T - T_c)P_z^2 + BP_z^4 - \gamma P_z\sigma_{xy} - \mu\sigma_{xy}^2 - E_zP_z - E_z^2/8\pi, \quad (19.20)$$

where for brevity we have put  $\gamma_{z,xy} = \gamma$ ,  $\mu_{xyxy} = \mu$ .‡ The terms involving the other components of  $\mathbf{P}$  and  $\sigma_{ik}$  are of no interest, since they lead to no anomaly of the piezoelectric properties near the transition point.

Equating to zero the derivative  $\partial\tilde{\Phi}/\partial P_z$  with  $E_z$  constant, we obtain

$$E_z = 2a(T - T_c)P_z + 4BP_z^3 - \gamma\sigma_{xy}. \quad (19.21)$$

The components of the strain tensor are found by differentiating the thermodynamic potential (19.20) with respect to the corresponding components  $\sigma_{ik}$  (see 17.4):§

$$u_{xy} = \frac{1}{2}\gamma P_z + \mu\sigma_{xy}. \quad (19.22)$$

In the non-pyroelectric phase when  $\mathbf{E}$  is small we can neglect the term in  $P_z^3$  in (19.21):

$$E_z = 2a(T - T_c)P_z - \gamma\sigma_{xy}.$$

Substituting  $P_z$  in (19.22), we find

$$u_{xy} = \frac{\gamma}{4a(T - T_c)}E_z + \left[ \mu + \frac{\gamma^2}{4a(T - T_c)} \right] \sigma_{xy}.$$

The coefficient of  $\sigma_{xy}$  in this formula represents the modulus of elasticity for deformations in which the field  $E_z$  is kept constant, while  $\mu$  in formula (19.22) is the modulus for constant

† The non-pyroelectric phase of a ferroelectric is piezoelectric if it belongs to one of eight out of the ten classes listed at the end of §17:  $D_2$ ,  $D_4$ ,  $D_{2d}$ ,  $S_4$ ,  $D_3$ ,  $D_6$ ,  $C_{3h}$ ,  $D_{3h}$ .

‡ Because the expansion is of a different type, the definitions of the tensors  $\gamma_{i,kj}$  and  $\mu_{iklm}$  are not the same as those denoted by the same letters in §17, but their symmetry properties are, of course, unchanged.

§ See the first footnote to §17, Problem 1, concerning differentiation with respect to the components of a tensor.

polarization  $P_z$ . Hence we can write

$$\mu^{(E)} = \mu^{(P)} + \gamma^2/4a(T - T_c), \quad (19.23)$$

where the superscripts indicate the nature of the deformation. We see that the two coefficients behave entirely differently: whereas  $\mu^{(P)}$  is a finite constant,  $\mu^{(E)}$  increases without limit as the transition point is approached.<sup>†</sup>

In the pyroelectric phase, formula (19.22) shows that the spontaneous polarization results in a certain deformation of the body. If there are no internal stresses and the field  $\mathbf{E}$  is zero, the strain  $u_{xy}$  is proportional to  $P_{z0}$ , i.e. varies with temperature as  $\sqrt{T_c - T}$ .

If the symmetry (cubic, for example) of the non-pyroelectric phase of a ferroelectric does not admit a linear piezoelectric effect, then the first non-vanishing terms in an expansion of the thermodynamic potential in powers of  $\sigma_{ik}$  and  $\mathbf{P}$  are quadratic in the components of  $\mathbf{P}$ , i.e. they are of the form

$$-\gamma_{iklm}P_iP_k\sigma_{lm}, \quad (19.24)$$

where  $\gamma_{iklm}$  is a tensor of rank four, symmetrical with respect to the pairs of suffixes  $i, k$  and  $l, m$ . In such cases, the strain in the pyroelectric phase due to the spontaneous polarization is quadratic in  $\mathbf{P}_0$ , and accordingly varies with temperature as  $T_c - T$ .

Doubt might be cast on the legitimacy of using the expression (19.24) in the thermodynamic potential, on the grounds that, as stated in §17, this potential can be used only when quadratic effects are neglected. However, the ferroelectrics form an exception because, near the transition point, the field  $\mathbf{E}$  is small compared with the polarization  $\mathbf{P}$  or induction  $\mathbf{D}$ , because of the unlimited increase in the dielectric susceptibility. The use of the thermodynamic potential involves the neglect of the quantities of the order of  $EDu_{ik}$  (or, what is the same thing,  $ED\sigma_{ik}$ ), whereas the expression (19.24) is of the order of  $D^2\sigma_{ik}$ .

## §20. Improper ferroelectrics

The theory of ferroelectrics given in §19 is based on identifying the polarization vector of the crystal with the order parameter which determines the change in the crystal symmetry in the phase transition. This is not always permissible, however. It may happen that the occurrence of spontaneous polarization does not in itself entirely determine the nature of the change in the crystal structure.

It is known (see SP 1, §145) that the order parameter in a second-order phase transition is a quantity, or a set of quantities, transformed by some irreducible (not the unit) representation of the symmetry group of the original ("symmetrical") phase. The transformation properties of the order parameter determine the nature of the change (the decrease) in the symmetry at the phase transition. The specific physical nature of the change is unimportant; the order parameter may be taken as any of various physical quantities, provided that they are related to one another by linear expressions and therefore have the same transformation properties.

The choice of the vector  $\mathbf{P}$  as the order parameter is equivalent to assuming that this is transformed by the same representation as the components of a (polar) vector. If the phase transition occurs with no change in the unit cell of the lattice (or with only a strain), the irreducible representations concerned are those of the point symmetry groups (the crystal

<sup>†</sup> The modulus  $\mu^{(D)} = \mu + \gamma^2/8\pi$ , which determines the strain for a constant induction  $D_z$ , is also a constant.

classes). In the biaxial classes (§13), each component of a vector is transformed by one of the one-dimensional representations. The same applies to a component of a vector along the principal axis of symmetry (threefold, fourfold or sixfold) in uniaxial crystals. For all these representations, the order parameter can be the corresponding component of the vector  $\mathbf{P}$ , and the theory based on the thermodynamic potential (19.1) is applicable to them. The components of  $\mathbf{P}$  in a plane perpendicular to the principal axis of symmetry in a uniaxial crystal are transformed by a two-dimensional irreducible representation and can serve as the order parameter for that representation. Lastly, in crystals with cubic symmetry, all three components of the vector are transformed by a single three-dimensional representation. This case corresponds to the theory of ferroelectricity based on the thermodynamic potential (19.14).

There are also, however, ferroelectric transitions in which the order parameter is transformed by an irreducible representation of the symmetrical phase which does not correspond to the components of a vector. In such cases, the order parameter is not the polarization but a physically different quantity; the spontaneous polarization arises as a secondary effect (it is assumed, of course, that the symmetry of the unsymmetrical phase allows pyroelectricity). These substances are called *improper ferroelectrics*; they differ considerably from ordinary ferroelectrics as regards the nature of the dielectric anomalies.† Here belong all ferroelectric transitions in which the unit cell changes, i.e. in which the translational symmetry of the lattice changes (the corresponding irreducible representations are certainly not realized by vector quantities invariant under translations)‡, but they may also be transitions without change of translational symmetry (the order parameter is transformed by an irreducible representation of the point group, not corresponding to the components of a vector).

In an ordinary ferroelectric transition, when the change in symmetry is entirely determined by the polarization vector, the transition is to a higher sub-group (that allows pyroelectricity) of the space group of the original (non-pyroelectric) phase. In an improper ferroelectric transition, the pyroelectric phase belongs to a sub-group of lower symmetry.

The specific thermodynamic properties of improper ferroelectrics may vary considerably, like the transformational properties of quantities that are transformed by different irreducible representations of the space groups. Let us now consider (again in terms of the Landau theory of phase transitions) one formal example to illustrate some important fundamental points.

We shall take a transition (without change in the unit cell) from a non-pyroelectric crystal of the class  $C_{3h}$  to the class  $C_1$ , which allows spontaneous polarization, the order parameter having two components  $\eta_1, \eta_2$  and being transformed by the irreducible representation  $E_u$  of the group  $C_{3h}$ ; the components  $P_x, P_y$  of the polarization vector in the plane perpendicular to the  $C_3$  axis being transformed by the representation  $E_g$ .

The thermodynamic potential  $\Phi$  near the transition point is to be expanded in powers of the order parameter  $\eta_1, \eta_2$  and the polarization  $P_x, P_y$ . For ferroelectricity to occur, there must be mixed invariants formed from the same quantities, and linear in the vector  $\mathbf{P}$ . There are two such invariants in this case: the real and imaginary parts of the product

† The possible existence of such ferroelectrics was noted by V. L. Indenbom (1960).

‡ All known improper ferroelectrics are in fact of this type.

$(\eta_1 + i\eta_2)^2 (P_x + iP_y)$ . We thus obtain an expansion in the form

$$\begin{aligned} \tilde{\Phi} = \Phi_0 + a(T - T_c)\eta^2 + B\eta^4 + \kappa\mathbf{P}^2 + C_1\eta^2 [P_x(\gamma_1^2 - \gamma_2^2) - 2P_y\gamma_1\gamma_2] \\ + C_2\eta^2 [P_y(\gamma_1^2 - \gamma_2^2) + 2P_x\gamma_1\gamma_2] - \mathbf{E} \cdot \mathbf{P} - \mathbf{E}^2/8\pi, \end{aligned} \quad (20.1)$$

where  $\eta^2 = \eta_1^2 + \eta_2^2$ ,  $\gamma_i = \eta_i/\eta$ ; the vectors  $\mathbf{E}$  and  $\mathbf{P}$  are in the  $xy$ -plane.

The order parameter and the polarization are determined by the condition that  $\tilde{\Phi}$  be a minimum (for constant  $\mathbf{E}$ ). Here we shall give only some characteristic results that are evident without actually making the calculation. The order parameter in the unsymmetrical phase is found to be proportional to  $(T_c - T)^{1/2}$ , as for any second-order transition in the Landau theory. The polarization arises as an effect of the second order in  $\eta$ , and is therefore proportional to  $T_c - T$ . The dielectric susceptibility does not tend to infinity as  $T \rightarrow T_c$  as in ordinary ferroelectrics, because it is not here determined by a coefficient of  $\eta^2$  that tends to zero. It does, however, have a finite discontinuity at the transition point. This is because, in the symmetrical phase, the order parameter  $\eta = 0$  and is not affected by the field  $\mathbf{E}$ ; in the unsymmetrical phase it does change, and this gives a further contribution to the susceptibility.

An improper ferroelectric transition is possible only when the order parameter has more than one component. With a one-component parameter  $\eta$ , the only possible mixed invariant linear in  $\mathbf{P}$  is  $\eta P_z$ , where  $P_z$  is one component of the vector  $\mathbf{P}$  (since  $\eta^2$  is an invariant for a one-dimensional representation). This would mean, however, that  $\eta$  and  $P_z$  had the same transformational properties, and therefore that  $P_z$  itself could be chosen as the order parameter.



## CHAPTER III

# STEADY CURRENT

### §21. The current density and the conductivity

LET US now consider the steady motion of charges in conductors, i.e. steady electric currents. We shall denote by  $\mathbf{j}$  the mean charge flux density or *electric current density*.† In a steady current, the spatial distribution of  $\mathbf{j}$  is independent of time, and satisfies the equation

$$\operatorname{div} \mathbf{j} = 0, \quad (21.1)$$

which states that the mean total charge in any volume of the conductor remains constant.

The electric field in the conductor in which a steady current flows is constant, and therefore satisfies the equation

$$\operatorname{curl} \mathbf{E} = 0, \quad (21.2)$$

i.e. it is a potential field.

Equations (21.1) and (21.2) must be supplemented by an equation relating  $\mathbf{j}$  and  $\mathbf{E}$ . This equation depends on the properties of the conductor, but in the great majority of cases it may be supposed linear (*Ohm's law*). If the conductor is homogeneous and isotropic, the linear relation is a simple proportionality:

$$\mathbf{j} = \sigma \mathbf{E}. \quad (21.3)$$

The coefficient  $\sigma$  depends on the nature and state of the conductor; it is called the *electrical conductivity*.

In a homogeneous conductor,  $\sigma = \text{constant}$  and, substituting (21.3) in (21.1), we have  $\operatorname{div} \mathbf{E} = 0$ . In this case the electric field potential satisfies Laplace's equation:  $\Delta \phi = 0$ .

At a boundary between two conducting media, the normal component of the current density must, of course, be continuous. Moreover, by the general condition that the tangential field component be continuous (which follows from  $\operatorname{curl} \mathbf{E} = 0$ ; cf. (1.7) and (6.9)), the ratio  $\mathbf{j}_t/\sigma$  must be continuous. Thus the boundary conditions on the current density are

$$j_{n1} = j_{n2}, \quad \mathbf{j}_{t1}/\sigma_1 = \mathbf{j}_{t2}/\sigma_2, \quad (21.4)$$

or, as conditions on the field,

$$\sigma_1 E_{n1} = \sigma_2 E_{n2}, \quad \mathbf{E}_{t1} = \mathbf{E}_{t2}. \quad (21.5)$$

† In this chapter we ignore the magnetic field due to the current, and therefore the reaction of that field on the current. If this effect is to be taken into account, the definition of the current density must be refined, which we do in §30.



At a boundary between a conductor and a non-conductor we have simply  $j_n = 0$ , or  $E_n = 0$ .†

An electric field in the presence of a current does mechanical work on the current-carrying particles moving in the conductor; the work done per unit time and volume is evidently equal to the scalar product  $\mathbf{j} \cdot \mathbf{E}$ . This work is dissipated into heat in the conductor. Thus the quantity of heat evolved per unit time and volume in a homogeneous conductor is

$$\mathbf{j} \cdot \mathbf{E} = \sigma E^2 = j^2 / \sigma. \quad (21.6)$$

This is *Joule's law*.‡

The evolution of heat results in an increase in the entropy of the body. When an amount of heat  $dQ = \mathbf{j} \cdot \mathbf{E} dV$  is evolved, the entropy of the volume element  $dV$  increases by  $dQ/T$ . The rate of change of the total entropy of the body is therefore

$$d\mathcal{S}/dt = \int (\mathbf{j} \cdot \mathbf{E}/T) dV. \quad (21.7)$$

Since the entropy must increase, this derivative must be positive. Putting  $\mathbf{j} = \sigma \mathbf{E}$ , we see that the conductivity  $\sigma$  must therefore be positive.

In an anisotropic body (a single crystal), the directions of the vectors  $\mathbf{j}$  and  $\mathbf{E}$  are in general different, and the linear relation between them is

$$j_i = \sigma_{ik} E_k, \quad (21.8)$$

where the quantities  $\sigma_{ik}$  form a tensor of rank two, the *conductivity tensor*, which is symmetrical (see below).

The following remark should be made here. The symmetry of the crystal would admit also an inhomogeneous term in the linear relation between  $\mathbf{j}$  and  $\mathbf{E}$ , giving  $j_i = \sigma_{ik} E_k + j_i^{(0)}$ , with  $\mathbf{j}^{(0)}$  a constant vector. The presence of this term would mean that the conductor was "pyroelectric", there being a non-zero field in it when  $\mathbf{j} = 0$ . In reality, however, this is impossible, because the entropy must increase: the term  $\mathbf{j}^{(0)} \cdot \mathbf{E}$  in the integrand in (21.7) could take either sign, and so  $d\mathcal{S}/dt$  could not be invariably positive.

Just as, for an isotropic medium,  $d\mathcal{S}/dt > 0$  leads to  $\sigma > 0$ , so for an anisotropic medium this condition means that the principal values of the tensor  $\sigma_{ik}$  must be positive.

The dependence of the number of independent components of the tensor  $\sigma_{ik}$  on the symmetry of the crystal is the same as for any symmetrical tensor of rank two (see §13): for biaxial crystals, all three principal values are different, for uniaxial crystals two are equal, and for cubic crystals all three are equal, i.e. a cubic crystal behaves as an isotropic body as regards its conductivity.

The symmetry of the conductivity tensor

$$\sigma_{ik} = \sigma_{ki} \quad (21.9)$$

is a consequence of the *symmetry of the kinetic coefficients*. This general principle, due to L. Onsager, may be conveniently formulated, for use here and in §§26–28, as follows (see SP 1, §120).

† It should be noticed that the equations  $\text{curl } \mathbf{E} = 0$ ,  $\text{div } (\sigma \mathbf{E}) = 0$  and the boundary conditions (21.5) thereon are formally identical with the equations for the electrostatic field in a dielectric, the only difference being that  $\epsilon$  is replaced by  $\sigma$ . This enables us to solve problems of the current distribution in an infinite conductor if the solutions of the corresponding electrostatic problems are known. When the conductor is bounded by a non-conductor this analogy does not serve, because in electrostatics there is no medium for which  $\epsilon = 0$ .

‡ In Russian "Joule and Lenz's law".

Let  $x_1, x_2, \dots$  be some quantities which characterize the state of the body at every point. We define also the quantities

$$X_a = -\partial S / \partial x_a, \quad (21.10)$$

where  $S$  is the entropy of unit volume of the body, and the derivative is taken at constant energy of the volume. In a state close to equilibrium, the quantities  $x_a$  are close to their equilibrium values, and the  $X_a$  are small. Processes will occur in the body which tend to bring it into equilibrium. The rates of change of the quantities  $x_a$  at each point are usually functions only of the values of the  $x_a$  (or  $X_a$ ) at that point. Expanding these functions in powers of  $X_a$  and taking only the linear terms, we have

$$\partial x_a / \partial t = - \sum_b \gamma_{ab} X_b. \quad (21.11)$$

Then we can assert that the coefficients  $\gamma_{ab}$  (the *kinetic coefficients*) are symmetrical with respect to the suffixes  $a$  and  $b$ :†

$$\gamma_{ab} = \gamma_{ba}. \quad (21.12)$$

In order to make practical use of this principle, it is necessary to choose the quantities  $x_a$  (or their derivatives  $\dot{x}_a$ ) in some manner, and then to determine the  $X_a$ . This can usually be done very simply by means of the formula for the rate of change of the total entropy of the body

$$\frac{d\mathcal{S}}{dt} = - \int \sum_a X_a \frac{\partial x_a}{\partial t} dV, \quad (21.13)$$

where the integration is extended over the whole volume of the body.

When a current flows in a conductor,  $d\mathcal{S}/dt$  is given by (21.7). Comparing this with (21.13), we see that, if the components of the current density vector  $\mathbf{j}$  are taken as the quantities  $\dot{x}_a$ , then the quantities  $X_a$  will be the components of the vector  $-\mathbf{E}/T$ . A comparison of formulae (21.8) and (21.11) shows that the kinetic coefficients in this case are the components of the conductivity tensor, multiplied by  $T$ . Thus the symmetry of this tensor follows immediately from the general relation (21.12).

## PROBLEMS

**PROBLEM 1.** A system of electrodes maintained at constant potentials  $\phi_a$  is immersed in a conducting medium. A current  $J_a$  flows from each electrode. Determine the total amount of Joule heat evolved in the medium per unit time.

**SOLUTION.** The required amount of heat  $Q$  is given by the integral

$$Q = \int \mathbf{j} \cdot \mathbf{E} dV = - \int \mathbf{j} \cdot \text{grad } \phi dV = - \int \text{div}(\phi \mathbf{j}) dV,$$

taken over the volume of the medium. We transform this into a surface integral, using the fact that  $j_n = 0$  at the outer boundary of the medium, while on the surfaces of the electrodes  $\phi = \text{constant} \equiv \phi_a$ . The result is  $Q = \sum \phi_a J_a$ .

**PROBLEM 2.** Determine the potential distribution in a conducting sphere with a current  $J$  entering at a point  $O$  and leaving at the point  $O'$  diametrically opposite to  $O$ .

**SOLUTION.** Near  $O$  and  $O'$  (Fig. 15) the potential must be of the forms  $\phi = J/2\pi\sigma R_1$  and  $\phi = -J/2\pi\sigma R_2$  respectively,  $R_1$  and  $R_2$  being the distances from  $O$  and  $O'$ . These functions satisfy Laplace's equation, and the

† It is assumed that  $x_a$  and  $x_b$  behave in the same way under time reversal.

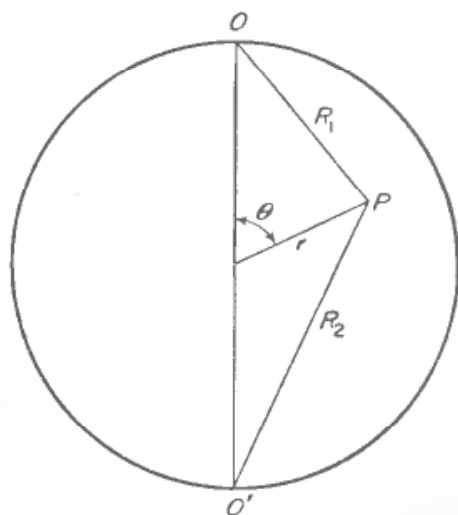


FIG. 15

integrals  $-\sigma \int \text{grad } \phi \cdot d\mathbf{f}$  over infinitesimal caps about  $O$  and  $O'$  are equal to  $\pm J$ . We seek the potential at an arbitrary point  $P$  in the sphere in the form

$$\phi = \frac{J}{2\pi\sigma} \left\{ \frac{1}{R_1} - \frac{1}{R_2} + \psi \right\},$$

where  $\psi$  is a solution of Laplace's equation having no poles in or on the sphere. It is evident from symmetry that  $\psi$ , like  $\phi$ , is a function of the spherical polar coordinates  $r$  and  $\theta$  only.

On the surface of the sphere ( $r = a$ ) we must have  $\partial\phi/\partial r = 0$ . Differentiating, we find the boundary condition on  $\psi$ :

$$\frac{\partial\psi}{\partial r} = \frac{1}{2a} \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \text{ for } r = a.$$

If  $f(r, \theta)$  is any solution of Laplace's equation, then the function

$$\int_0^r \frac{f(r, \theta)}{r} dr$$

is also a solution.† Comparing this with the above boundary condition, we see that the condition is met by the solution

$$\psi = \frac{1}{2} \int_0^r \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \frac{dr}{r}.$$

Substituting  $R_{1,2} = \sqrt{(a^2 + r^2 \mp 2ar \cos \theta)}$  and effecting the integration, we have finally

$$\phi = \frac{J}{2\pi\sigma} \left\{ \frac{1}{R_1} - \frac{1}{R_2} + \frac{1}{2a} \left( \sinh^{-1} \frac{a+r \cos \theta}{r \sin \theta} - \sinh^{-1} \frac{a-r \cos \theta}{r \sin \theta} \right) \right\},$$

if  $\phi = 0$  when  $r = 0$ .

**PROBLEM 3.** Show that the current distribution in a conductor is such that the energy dissipated is a minimum.

† This is easily seen either by direct calculation or from the fact that any solution  $f(r, \theta)$  of Laplace's equation depending only on  $r$  and  $\theta$  can be written  $f = \sum c_n r^n P_n(\cos \theta)$ , where the  $c_n$  are constants and the  $P_n$  are Legendre Polynomials.

**SOLUTION.** The minimum concerned is that of the integral  $\int \mathbf{j} \cdot \mathbf{E} dV = \int (j^2/\sigma) dV$ , with the subsidiary condition  $\text{div } \mathbf{j} = 0$  (conservation of charge). Varying with respect to  $\mathbf{j}$  the integral  $\int [(j^2/\sigma) - 2\phi \text{div } \mathbf{j}] dV$ , where  $2\phi$  is an undetermined Lagrange multiplier, and equating the result to zero, we obtain the equation  $\mathbf{j} = -\sigma \text{grad } \phi$  or  $\text{curl } (\mathbf{j}/\sigma) = 0$ , which is the same as (21.2) and (21.3).

## §22. The Hall effect

If a conductor is in an external magnetic field  $\mathbf{H}$ , the relation between the current density and the electric field is again given by  $j_i = \sigma_{ik} E_k$ , but the components of the conductivity tensor  $\sigma_{ik}$  are functions of  $\mathbf{H}$  and, what is particularly important, they are no longer symmetrical with respect to the suffixes  $i$  and  $k$ . The symmetry of this tensor was proved in §21 from the symmetry of the kinetic coefficients. In a magnetic field, however, this principle must be formulated somewhat differently: when the suffixes are interchanged, the direction of the magnetic field must be reversed (see SP 1, §120). Hence we now have for the components  $\sigma_{ik}(\mathbf{H})$  the relations

$$\sigma_{ik}(\mathbf{H}) = \sigma_{ki}(-\mathbf{H}). \quad (22.1)$$

The quantities  $\sigma_{ik}(\mathbf{H})$  and  $\sigma_{ki}(\mathbf{H})$  are not equal.

Like any tensor of rank two,  $\sigma_{ik}$  can be divided into symmetrical and antisymmetrical parts, which we denote by  $s_{ik}$  and  $a_{ik}$ :

$$\sigma_{ik} = s_{ik} + a_{ik}. \quad (22.2)$$

By definition

$$s_{ik}(\mathbf{H}) = s_{ki}(\mathbf{H}), \quad a_{ik}(\mathbf{H}) = -a_{ki}(\mathbf{H}), \quad (22.3)$$

and from (22.1) it follows that

$$\left. \begin{aligned} s_{ik}(\mathbf{H}) &= s_{ki}(-\mathbf{H}), \\ a_{ik}(\mathbf{H}) &= a_{ki}(-\mathbf{H}) = -a_{ik}(\mathbf{H}). \end{aligned} \right\} \quad (22.4)$$

Thus the components of the tensor  $s_{ik}$  are even functions of the magnetic field, and those of  $a_{ik}$  are odd functions.

Any antisymmetrical tensor  $a_{ik}$  of rank two corresponds to some axial vector, whose components are

$$a_x = a_{yz}, \quad a_y = -a_{xz}, \quad a_z = a_{xy}. \quad (22.5)$$

In terms of this vector, the components of the product  $a_{ik} E_k$  can be written as those of the vector product  $\mathbf{E} \times \mathbf{a}$ :

$$j_i = \sigma_{ik} E_k = s_{ik} E_k + (\mathbf{E} \times \mathbf{a})_i, \quad (22.6)$$

The Joule heat generated by the passage of the current is given by the product  $\mathbf{j} \cdot \mathbf{E}$ . Since the vectors  $\mathbf{E} \times \mathbf{a}$  and  $\mathbf{E}$  are perpendicular, their scalar product is zero identically, and so

$$\mathbf{j} \cdot \mathbf{E} = s_{ik} E_i E_k, \quad (22.7)$$

i.e. the Joule heat is determined (for a given field  $\mathbf{E}$ ) only by the symmetrical part of the conductivity tensor.

If the external magnetic field is sufficiently weak, the components of the conductivity tensor may be expanded in powers of that field. Since the function  $\mathbf{a}(\mathbf{H})$  is odd, the

expansion of this vector will involve only odd powers. The first terms are linear in the field, i.e. they are of the form

$$a_i = \alpha_{ik} H_k. \quad (22.8)$$

The vectors  $\mathbf{a}$  and  $\mathbf{H}$  are both axial, and the constants  $\alpha_{ik}$  therefore form an ordinary (polar) tensor. The expansion of the even functions  $s_{ik}(\mathbf{H})$  will involve only even powers. The first term is the conductivity  $\sigma_{ik}^{(0)}$  in the absence of the field, and the next terms are quadratic in the field:

$$s_{ik} = \sigma_{ik}^{(0)} + \beta_{iklm} H_l H_m. \quad (22.9)$$

The tensor  $\beta_{iklm}$  is symmetrical with respect to  $i, k$  and  $l, m$ .

Thus the principal effect of the magnetic field is linear in the field and is given by the term  $\mathbf{E} \times \mathbf{a}$ ; it is called the *Hall effect*. As we see, it gives rise to a current perpendicular to the electric field, whose magnitude is proportional to the magnetic field. It should be borne in mind, however, that, for an arbitrary anisotropic medium, the Hall current is not the only current perpendicular to  $\mathbf{E}$ ; the current  $s_{ik} E_k$  also has a component in such a direction.

The Hall effect may be differently regarded if we use the inverse formulae which express  $\mathbf{E}$  in terms of the current density:  $E_i = \sigma^{-1}_{ik} j_k$ . The inverse tensor  $\sigma^{-1}_{ik}$ , like  $\sigma_{ik}$  itself, can be resolved into a symmetrical part  $\rho_{ik}$  and an antisymmetrical part which may be represented by an axial vector  $\mathbf{b}$ :

$$E_i = \rho_{ik} j_k + (\mathbf{j} \times \mathbf{b})_i. \quad (22.10)$$

The tensor  $\rho_{ik}$  and the vector  $\mathbf{b}$  have the same properties as  $s_{ik}$  and  $\mathbf{a}$ . In particular, in weak magnetic fields the vector  $\mathbf{b}$  is linear in the field. In formula (22.10) the Hall effect is represented by the term  $\mathbf{j} \times \mathbf{b}$ , i.e. by an electric field perpendicular to the current and proportional to the magnetic field and to the current  $\mathbf{j}$ .

The above relations are much simplified if the conductor is isotropic. The vectors  $\mathbf{a}$  and  $\mathbf{b}$  must then be parallel to the magnetic field, by symmetry. The only non-zero components of the tensor  $\rho_{ik}$  are  $\rho_{xx} = \rho_{yy} = \rho_{zz}$ , the field being in the  $z$ -direction. Denoting these two quantities by  $\rho_{\perp}$  and  $\rho_{\parallel}$  and taking the current to lie in the  $xz$ -plane, we have

$$E_x = \rho_{\perp} j_x, \quad E_y = -b j_x, \quad E_z = \rho_{\parallel} j_z. \quad (22.11)$$

Hence we see that, in an isotropic conductor, the Hall field is the only electric field which is perpendicular to both the current and the magnetic field.

In weak magnetic fields, the vectors  $\mathbf{b}$  and  $\mathbf{H}$  are related (in an isotropic body) simply by

$$\mathbf{b} = -R\mathbf{H}. \quad (22.12)$$

The constant  $R$  (called the *Hall constant*) may be either positive or negative. The form of the terms quadratic in  $\mathbf{H}$  in the relation between  $\mathbf{E}$  and  $\mathbf{j}$ , which enter through the tensor  $\rho_{ik}$ , is easily seen from the fact that the only vectors linear in  $\mathbf{j}$  and quadratic in  $\mathbf{H}$  which can be constructed from  $\mathbf{j}$  and  $\mathbf{H}$  are  $(\mathbf{j} \cdot \mathbf{H})\mathbf{H}$  and  $H^2\mathbf{j}$ . Hence the general form of the relation between  $\mathbf{E}$  and  $\mathbf{j}$  in an isotropic body, as far as the terms quadratic in  $\mathbf{H}$ , is

$$\mathbf{E} = \rho^{(0)}\mathbf{j} + R\mathbf{H} \times \mathbf{j} + \beta_1 H^2\mathbf{j} + \beta_2 (\mathbf{j} \cdot \mathbf{H})\mathbf{H}. \quad (22.13)$$

### PROBLEM

Express the components of the inverse tensor  $\sigma^{-1}_{ik}$  in terms of those of  $s_{ik}$  and  $\mathbf{a}$ .

**SOLUTION.** The calculations are most simply effected by taking a system of coordinates in which the axes are the principal axes of the tensor  $s_{ik}$ ; the form of the results in an arbitrary coordinate system can easily be deduced

from their form in this particular case. The determinant  $|\sigma|$  is

$$|\sigma| = \begin{vmatrix} s_{xx} & a_z & -a_y \\ -a_z & s_{yy} & a_x \\ a_y & -a_x & s_{zz} \end{vmatrix} \\ = s_{xx}s_{yy}s_{zz} + s_{xx}a_x^2 + s_{yy}a_y^2 + s_{zz}a_z^2.$$

In the general case we evidently have

$$|\sigma| = |s| + s_{ik}a_i a_k.$$

From the minors of this determinant we find the components of the inverse tensor:

$$\sigma^{-1}_{xx} = \rho_{xx} = (s_{yy}s_{zz} + a_x^2)/|\sigma|,$$

$$\sigma^{-1}_{xy} = \rho_{xy} + b_z = (a_x a_y - a_z s_{zz})/|\sigma|, \dots$$

The general expressions which give these results for the particular system of coordinates chosen are

$$\rho_{ik} = \{s^{-1}_{ik}|s| + a_i a_k\}/|\sigma|, \quad b_i = -s_{ik}a_k/|\sigma|.$$

This completes the solution.

### §23. The contact potential

In order to remove a charged particle through the surface of a conductor, work must be done. The work required for a thermodynamically reversible removal of the particle is called the *work function*. This quantity is always positive; this follows immediately from the fact that a point charge is attracted to any neutral body, and therefore to any conductor (see §14). The work in question will be denoted by  $eW$ , where  $e$  is the charge on the particle; the sign of the *work potential*  $W$  thus defined is the same as that of the charge on the particle removed.

The work function depends both on the nature of the conductor (and its thermodynamic state, i.e. its temperature and density) and on that of the charged particle. For example, the work function for a given metal is different for the removal of a conduction electron and for the removal of an ion from the surface. It must also be emphasized that the work function is characteristic of the surface of the conductor. It therefore depends, for instance, on the treatment of the surface and the "contamination" of it. If the conductor is a single crystal, then the work function is different for different faces.

To ascertain the physical nature of the dependence of the work function on the properties of the surface, let us establish its relation to the electric structure of the surface layer. If  $\rho(x)$  is the charge density *not* averaged over physically infinitesimal segments of the  $x$ -axis (perpendicular to the layer), we can write Poisson's equation in the layer as  $d^2\phi/dx^2 = -4\pi\rho$ . Let the conductor occupy the region  $x < 0$ . Then a first integration gives

$$\frac{d\phi}{dx} = -4\pi \int_{-\infty}^x \rho dx,$$

and a second integration (by parts) gives

$$\phi - \phi(-\infty) = -4\pi x \int_{-\infty}^x \rho dx + 4\pi \int_{-\infty}^x x\rho dx.$$

For  $x \rightarrow \infty$ , the integral

$$\int_{-\infty}^x \rho dx$$

tends very rapidly to zero (since the surface of an uncharged conductor is electrically neutral). Hence

$$\phi(+\infty) - \phi(-\infty) = 4\pi \int_{-\infty}^{\infty} x\rho dx.$$

The integral on the right is the dipole moment of the charges near the surface of the body. These charges form a "double layer", in which charges of opposite sign are separated and the dipole moment is non-zero. The structure of the double layer, of course, depends on the properties of the surface (its crystallographic direction, contamination, etc.). The difference in the work potential for different surfaces of a given conductor is determined by the difference in the dipole moments.

If two different conductors are placed in contact, an exchange of charged particles may occur between them. Charges pass from the body with the smaller work function to that with the greater until a potential difference between them is set up which prevents further movement of charge. This is called a *contact potential*.

Fig. 16 shows a cross-section of two conductors in contact ( $a$  and  $b$ ) near their surfaces  $AO$  and  $OB$ . Let the potentials of these surfaces be  $\phi_a$  and  $\phi_b$  respectively. Then the contact potential is  $\phi_{ab} = \phi_b - \phi_a$ . The quantitative relation between this potential and the work functions is given by the condition of thermodynamic equilibrium. Let us consider the work which must be done on a particle with charge  $e$  to remove it from the conductor  $a$  through the surface  $AO$ , transfer it to the surface  $OB$ , and finally carry it into the conductor  $b$ . In a state of thermodynamic equilibrium, this work must be zero.† The work done on the particle in the three stages mentioned is  $eW_a$ ,  $e(\phi_b - \phi_a)$ , and  $-eW_b$  respectively. Putting the sum of these equal to zero, we find the required relation:

$$\phi_{ab} = W_b - W_a. \quad (23.1)$$

Thus the contact potential of the neighbouring free surfaces of two conductors in contact is equal to the difference in their work functions.

The existence of the contact potential results in the appearance of an electric field in the space outside the conductors. It is easy to determine this field near the line of contact of the surfaces. In a small region near this line (the point  $O$  in Fig. 16), the surfaces may be

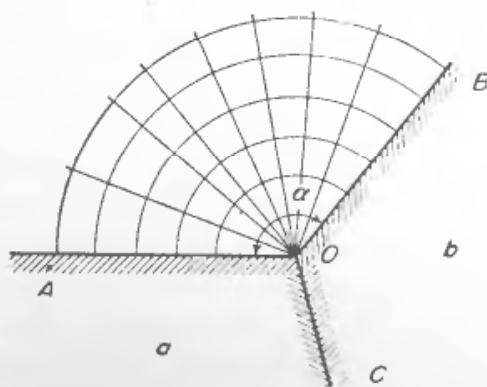


FIG. 16

† Of course, in reality a particle can pass from one conductor to another only through their surface of contact, and not through the space adjoining them, but the work done is independent of the path.

regarded as plane. The field potential outside the conductors satisfies the equation

$$\Delta \phi \equiv \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0,$$

where  $r$  and  $\theta$  are polar coordinates with origin at  $O$ ; on  $AO$  and  $OB$  the potential takes given constant values. We are interested in the solution which contains the lowest power of  $r$ ; this is the leading term in an expansion of the potential in powers of the small distance  $r$ . The solution concerned is  $\phi = \text{constant} \times \theta$ . Measuring the angle  $\theta$  from  $AO$  and arbitrarily taking the potential on  $AO$  as zero, we have

$$\phi = \phi_{ab} \theta / \alpha, \quad (23.2)$$

where  $\alpha$  is the angle  $AOB$ . Thus the equipotential lines in the plane of the diagram are straight lines diverging from  $O$ . The lines of force are arcs of circles centred at  $O$ . The field is

$$E = -\frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\phi_{ab}}{\alpha} \frac{1}{r}; \quad (23.3)$$

it decreases inversely as the distance from  $O$ .

As has been said above, "contact" potentials also exist between the various faces of a single crystal of metal. Hence an electric field of the kind just described must exist near the edges of the crystal.†

If several metallic conductors (at equal temperatures) are connected together, the potential between the extreme conductors is, as we easily deduce from formula (23.1), simply the difference of their work potentials, as it is for two conductors in direct contact. In particular, if the metal at each end is the same, the contact potential between the ends is zero. This is evident, however, because if there were a potential difference between two like conductors, a current would flow when they were connected, in contradiction to the second law of thermodynamics.

## §24. The galvanic cell

The statement at the end of §23 ceases to be valid if the circuit includes conductors in which the current is carried by different means (e.g. metals and solutions of electrolytes). Because the work function of a conductor is different for different charged particles (electrons and ions), the total contact potential in the circuit is not zero even when the conductors at each end are similar. This total potential difference is called the *electromotive force* or *e.m.f.* in the circuit; it is just the potential difference between the two like conductors before the circuit is closed. When the circuit is closed, a current flows in it; this is the basis of the operation of what are called *galvanic cells*. The energy which maintains the current in the circuit is supplied by chemical transformations occurring in the cell.

When we go completely round any closed circuit the field potential must, of course, return to its original value, i.e. the total change in the potential must be zero. Let us consider, for example, a contour on the surface of the conductors. When we pass from one conductor to another, the potential has a discontinuity  $\phi_{ab}$ . The potential drop across any

† In reality, all such fields are usually compensated by the field of ions from the atmosphere which "adhere" to the surface of the crystal.



conductor is  $RJ$ , where  $J$  is the total current flowing through it and  $R$  is its resistance. Hence the total change in the potential round the circuit is  $\Sigma\phi_{ab} - \Sigma JR$ . Putting this equal to zero and using the facts that  $J$  is the same at every point in the circuit and  $\Sigma\phi_{ab}$  is the electromotive force  $\mathcal{E}$ , we find

$$J\Sigma R = \mathcal{E}, \quad (24.1)$$

so that the current in a circuit containing a galvanic cell is equal to the e.m.f. divided by the total resistance of all the conductors in the circuit (including, of course, the internal resistance of the cell itself).

Although the e.m.f. of a galvanic cell can be expressed as a sum of contact potentials, it is very important to note that it is in reality a thermodynamic quantity, determined entirely by the states of the conductors and independent of the properties of the surfaces separating them. This is clear, because  $\mathcal{E}$  is just the work per unit charge which must be done on a charged particle when it is carried reversibly along the closed circuit.

To illustrate this, let us consider a galvanic cell consisting of two electrodes of metals  $A$  and  $B$  immersed in solutions of electrolytes  $AX$  and  $BX$ ,  $X^-$  being any anion. Let  $\zeta_A$  and  $\zeta_B$  be the chemical potentials of the metals  $A$  and  $B$ , and  $\zeta_{AX}$  and  $\zeta_{BX}$  those of the electrolytes in solution.† If an elementary charge  $e$  is carried along the closed circuit, an ion  $A^+$  passes into solution from the electrode  $A$  and an ion  $B^+$  passes out of solution to the electrode  $B$ , the change in the charges on the electrodes being compensated by the passage of an electron from  $A$  to  $B$  through the external circuit. The result is that the electrode  $A$  loses one neutral atom, the electrode  $B$  gains one, and in the electrolyte solution one molecule of  $BX$  is replaced by one of  $AX$ . Since the work done in a reversible process (at constant temperature and pressure) is equal to the change in the thermodynamic potential of the system, we have

$$e\mathcal{E}_{AB} = (\zeta_B - \zeta_{BX}) - (\zeta_A - \zeta_{AX}), \quad (24.2)$$

which expresses the e.m.f. of the cell in terms of the properties of the material of the electrodes and of the electrolyte solution.

From (24.2) we can also draw the following conclusion. If the solution contains three electrolytes  $AX$ ,  $BX$ ,  $CX$  and three metallic electrodes  $A$ ,  $B$ ,  $C$ , then the e.m.f.s between each pair of them are related by

$$\mathcal{E}_{AB} + \mathcal{E}_{BC} = \mathcal{E}_{AC}. \quad (24.3)$$

Using the general formulae of thermodynamics, we can relate the e.m.f. of a galvanic cell to the heat evolved when a current flows, which of course is actually an irreversible phenomenon. Let  $Q$  be the amount of heat generated (both in the cell itself and in the external circuit) when the unit charge passes along the circuit;  $Q$  is just the heat of the reaction which occurs in the cell when a current flows. By a well-known formula of thermodynamics (see *SP* 1, §91), it is related to the work  $\mathcal{E}$  by

$$Q = -T^2 \frac{\partial}{\partial T} \left( \frac{\mathcal{E}}{T} \right). \quad (24.4)$$

The definition of the partial derivative with respect to temperature depends on the conditions under which the process occurs. For example, if the current flows at constant pressure (as usually happens), then the differentiation is effected at constant pressure.

† In this section we use the ordinary chemical potentials, i.e. those defined with respect to one particle.

### §25. Electrocapillarity

The presence of charges on the boundary between two conducting media affects the surface tension there. This phenomenon is called *electrocapillarity*. In practice, the media concerned are both liquids; usually one is a liquid metal (mercury) and the other is a solution of an electrolyte.

Let  $\phi_1, \phi_2$  be the potentials of the two conductors, and  $e_1, e_2$  the charges at the surface of separation. These charges are equal in magnitude and opposite in sign, and thus form a double layer on the surface.

The differential of the potential  $\tilde{g}\phi$  of a system of two conductors at given temperature and pressure is, taking into account the surface of separation,

$$d\tilde{g}\phi = \alpha dS - e_1 d\phi_1 - e_2 d\phi_2, \quad (25.1)$$

where the term  $\alpha dS$  is the work done in a reversible change  $dS$  in the area  $S$  of the surface of separation;  $\alpha$  is the surface-tension coefficient (see SP 1, §154).

The thermodynamic potential  $\tilde{g}\phi$  in (25.1) may be replaced by its "surface part"  $\tilde{g}\phi_s$ , since the volume part is constant for given temperature and pressure, and is therefore of no interest here. Putting  $e_1 = -e_2 \equiv e$  and the potential difference  $\phi_1 - \phi_2 = \phi$ , we can write (25.1) as

$$d\tilde{g}\phi_s = \alpha dS - e d\phi. \quad (25.2)$$

Hence

$$(\partial \tilde{g}\phi_s / \partial S)_\phi = \alpha, \quad (25.3)$$

$\alpha$  being expressed as a function of  $\phi$ . Integrating, we find that  $\tilde{g}\phi_s = \alpha S$ . Substitution in (25.2) gives  $d(\alpha S) = \alpha dS - e d\phi$ , or  $S d\alpha = -e d\phi$ , whence

$$\sigma = -(\partial \alpha / \partial \phi)_{P,T}, \quad (25.4)$$

where  $\sigma = e/S$  is the charge per unit area of the surface. The relation (25.4), derived by G. Lippmann and J. W. Gibbs, is the fundamental formula in the theory of electrocapillarity.

In a state of equilibrium, the thermodynamic potential  $\tilde{g}\phi$  must be a minimum for given values of the electric potentials on the conductors. Regarding it as a function of the surface charges  $e$ , we can write the necessary conditions for a minimum as

$$\partial \tilde{g}\phi_s / \partial e = 0, \quad \partial^2 \tilde{g}\phi_s / \partial e^2 > 0, \quad (25.5)$$

where the derivatives are taken at constant area  $S$ . To calculate these, we express  $\tilde{g}\phi_s$  in terms of the thermodynamic potential  $g\phi_s = g\phi_s(e)$ :

$$\tilde{g}\phi_s = g\phi_s(e) - e_1 \phi_1 - e_2 \phi_2 = g\phi_s(e) - e\phi. \quad (25.6)$$

The vanishing of the first derivative gives

$$\frac{\partial \tilde{g}\phi_s}{\partial e} = \frac{\partial g\phi_s}{\partial e} - \phi = 0,$$

and then the condition for the second derivative to be positive becomes

$$\frac{\partial^2 \tilde{g}\phi_s}{\partial e^2} = \frac{\partial^2 g\phi_s}{\partial e^2} = \frac{\partial \phi}{\partial e} = \frac{1}{S} \frac{\partial \phi}{\partial \sigma} > 0,$$

or

$$\partial \sigma / \partial \phi > 0. \quad (25.7)$$

This result was to be expected, since the double layer on the surface may be regarded as a capacitor with capacitance  $\partial e/\partial\phi$ .

Differentiating equation (25.4) with respect to  $\phi$  and using (25.7), we find that

$$\partial^2\alpha/\partial\phi^2 < 0. \quad (25.8)$$

This means that the point where  $\partial\alpha/\partial\phi = -\sigma = 0$  is a maximum of  $\alpha$  as a function of  $\phi$ .

## §26. Thermoelectric phenomena

The condition that there should be no current in a metal is that there is thermodynamic equilibrium with respect to the conduction electrons. This means not only that the temperature must be constant throughout the body, but also that the sum  $e\phi + \zeta_0$  should be constant, where  $\zeta_0$  is the chemical potential of the conduction electrons in the metal (for  $\phi = 0$ ).† If the metal is not homogeneous,  $\zeta_0$  is not constant throughout the body even if the temperature is constant. Hence the constancy of the electric potential  $\phi$  in this case does not mean the absence of a current in the metal, although the field  $\mathbf{E} = -\text{grad } \phi$  is zero. This makes the ordinary definition of  $\phi$  (as the average of the true potential) inconvenient, if we wish to take inhomogeneous conductors into consideration.

It is natural to redefine the potential as  $\phi + \zeta_0/e$ , and we shall write this henceforward as  $\phi$  simply.‡ In a homogeneous metal, the change amounts to the adding of an unimportant constant to the potential. Accordingly, the "field"  $\mathbf{E} = -\text{grad } \phi$  (which we shall use henceforward) is the same as the true mean field only in a homogeneous metal, and in general the two differ by the gradient of some function of the state.§

With this definition, the current and field are both zero in a state of thermodynamic equilibrium with respect to the conduction electrons, and the relation between them is  $\mathbf{j} = \sigma\mathbf{E}$  (or  $j_i = \sigma_{ik}E_k$ ) even if the metal is not homogeneous.

Let us now consider a non-uniformly heated metal, which cannot be in thermodynamic equilibrium (with respect to the electrons). Then the field  $\mathbf{E}$  is not zero even if the current is zero. In general, when both the current density  $\mathbf{j}$  and the temperature gradient  $\text{grad } T$  are not zero, the relation between these quantities and the field can be written

$$\mathbf{E} = \mathbf{j}/\sigma + \alpha\text{grad } T. \quad (26.1)$$

Here  $\sigma$  is the ordinary conductivity, and  $\alpha$  is another quantity which is an electrical characteristic of the metal. Here we suppose for simplicity that the substance is isotropic (or of cubic symmetry), and therefore write the proportionality coefficients as scalars. The linear relation between  $\mathbf{E}$  and  $\text{grad } T$  is, of course, merely the first term of an expansion, but it is sufficient in view of the smallness of the temperature gradients occurring in practice.

The same formula (26.1), in the form

$$\mathbf{j} = \sigma(\mathbf{E} - \alpha\text{grad } T), \quad (26.2)$$

† See SP 1, §25. Here we take  $\zeta$  to be the chemical potential defined in the usual manner, viz. per unit particle (electron).

‡ This definition can also be formulated as follows: the new  $e\phi$  is the change in the free energy when one electron is isothermally brought into the metal. In other words,  $\phi = \partial F/\partial\rho$ , where  $F$  is the free energy of the metal and  $\rho$  the charge on the conduction electrons per unit volume.

§ It must be emphasized that  $e\mathbf{E}$  is then not the force on the charge  $e$ . Consequently, this definition of  $\mathbf{E}$ , which is suitable in a phenomenological theory, may be inconvenient in the microscopic theory when calculating the kinetic coefficients (cf. PK, §44).

shows that a current can flow in a non-uniformly heated metal even if the field  $\mathbf{E}$  is zero.

As well as the electric current density  $\mathbf{j}$ , we can consider the energy flux density  $\mathbf{q}$ . First of all, this quantity contains an amount  $\phi\mathbf{j}$  resulting simply from the fact that each charged particle (electron) carries with it an energy  $e\phi$ . The difference  $\mathbf{q} - \phi\mathbf{j}$ , however, does not depend on the potential, and can be generally written as a linear function of the gradients  $\mathbf{grad}\phi = -\mathbf{E}$  and  $\mathbf{grad}T$ , similarly to formula (26.2) for the current density. We shall for the present write this as

$$\mathbf{q} - \phi\mathbf{j} = \beta\mathbf{E} - \gamma\mathbf{grad}T.$$

The symmetry of the kinetic coefficients gives a relation between the coefficient  $\beta$  and the coefficient  $\alpha$  in (26.2). To derive this, we calculate the rate of change of the total entropy of the conductor. The amount of heat evolved per unit time and volume is  $-\text{div}\mathbf{q}$ . Hence we can put

$$\frac{d\mathcal{S}}{dt} = - \int \frac{\text{div}\mathbf{q}}{T} dV.$$

Using the equation  $\text{div}\mathbf{j} = 0$ , we have

$$\frac{\text{div}\mathbf{q}}{T} = \frac{1}{T} \{ \text{div}(\mathbf{q} - \phi\mathbf{j}) + \text{div}\phi\mathbf{j} \} = \frac{1}{T} \text{div}(\mathbf{q} - \phi\mathbf{j}) - \frac{\mathbf{E} \cdot \mathbf{j}}{T}.$$

The first term is integrated by parts, giving

$$\frac{d\mathcal{S}}{dt} = \int \frac{\mathbf{E} \cdot \mathbf{j}}{T} dV - \int \frac{(\mathbf{q} - \phi\mathbf{j}) \cdot \mathbf{grad}T}{T^2} dV. \quad (26.3)$$

This formula shows that, if we take as the quantities  $\partial x_a / \partial t$  (see §21) the components of the vectors  $\mathbf{j}$  and  $\mathbf{q} - \phi\mathbf{j}$ , then the corresponding quantities  $X_a$  are the components of the vectors  $-\mathbf{E}/T$  and  $\mathbf{grad}T/T^2$ . Accordingly in the relations

$$\mathbf{j} = \sigma T \frac{\mathbf{E}}{T} - \sigma \alpha T^2 \frac{\mathbf{grad}T}{T^2},$$

$$\mathbf{q} - \phi\mathbf{j} = \beta T \frac{\mathbf{E}}{T} - \gamma T^2 \frac{\mathbf{grad}T}{T^2},$$

the coefficients  $\sigma \alpha T^2$  and  $\beta T$  must be equal. Thus  $\beta = \sigma \alpha T$ , so that  $\mathbf{q} - \phi\mathbf{j} = \sigma \alpha T \mathbf{E} - \gamma \mathbf{grad}T$ . Finally, expressing  $\mathbf{E}$  in terms of  $\mathbf{j}$  and  $\mathbf{grad}T$  by (26.1), we have the result

$$\mathbf{q} = (\phi + \alpha T)\mathbf{j} - \kappa \mathbf{grad}T, \quad (26.4)$$

where  $\kappa = \gamma - T\alpha^2\sigma$  is simply the ordinary thermal conductivity, which gives the heat flux in the absence of an electric current.

It should be pointed out that the condition that  $d\mathcal{S}/dt$  should be positive places no new restriction on the thermoelectric coefficients. Substituting (26.1) and (26.4) in (26.3), we obtain

$$\frac{d\mathcal{S}}{dt} = \int \left( \frac{j^2}{\sigma T} + \frac{\kappa (\mathbf{grad}T)^2}{T^2} \right) dV > 0, \quad (26.5)$$

whence we find only that the coefficients of thermal and electrical conductivity must be positive.

In the above formulae it was tacitly assumed that an inhomogeneity of pressure (or

density) at constant temperature cannot cause a field (or current) to appear in the conductor, and consequently no term in  $\text{grad } p$  was included in (26.2) or (26.4). The existence of such terms would, in fact, contradict the law of the increase of entropy: the integrand in (26.5) would then contain terms in the products  $\mathbf{j} \cdot \text{grad } p$  and  $\text{grad } T \cdot \text{grad } p$ , which could be of either sign, and so the integral could not be necessarily positive.

The relations (26.1) and (26.4) indicate various thermoelectric effects. Let us consider the amount of heat  $-\text{div } \mathbf{q}$  evolved per unit time and volume in the conductor. Taking the divergence of (26.4), we have

$$\begin{aligned} Q &= -\text{div } \mathbf{q} \\ &= \text{div} (\kappa \text{ grad } T) + \mathbf{E} \cdot \mathbf{j} + \mathbf{j} \cdot \text{grad} (\alpha T), \end{aligned}$$

or, substituting (26.1),

$$Q = \text{div} (\kappa \text{ grad } T) + \frac{j^2}{\sigma} - T \mathbf{j} \cdot \text{grad } \alpha. \quad (26.6)$$

The first term on the right pertains to ordinary thermal conduction, and the second term, proportional to the square of the current, is the Joule heat. The term of interest here is the third, which gives the thermoelectric effects.

Let us assume the conductor to be homogeneous. Then the change in  $\alpha$  is due only to the temperature gradient, and  $\text{grad } \alpha = (d\alpha/dT) \text{ grad } T$ ; if, as usually happens, the pressure is constant through the body,  $d\alpha/dT$  must be taken as  $(\partial\alpha/\partial T)_p$ . Thus the amount of heat evolved (called the *Thomson effect*) is

$$\rho \mathbf{j} \cdot \text{grad } T, \text{ where } \rho = -T d\alpha/dT. \quad (26.7)$$

The coefficient  $\rho$  is called the *Thomson coefficient*. It should be noticed that this effect is proportional to the first power of the current, and not to the second power like the Joule heat. It therefore changes sign when the current is reversed. The coefficient  $\rho$  may be either positive or negative. If  $\rho > 0$ , the Thomson heat is positive (i.e. heat is emitted) when the current flows in the direction of increasing temperature, and heat is absorbed when it flows in the opposite direction; if  $\rho < 0$  the reverse is true.

Another thermal effect, called the *Peltier effect*, occurs when a current passes through a junction of two different metals. At the surface of contact, the temperature, the potential and the normal components of the current density and energy flux density are all continuous. Denoting by the suffixes 1 and 2 the values of quantities for the two metals and equating the normal components of  $\mathbf{q}$  (26.4) on the two sides, we have, since  $\phi$ ,  $T$  and  $j_x$  are continuous,

$$[-\kappa \partial T / \partial x]_1^2 = -j_x T (\alpha_2 - \alpha_1),$$

the  $x$ -axis being taken along the normal to the surface. If the positive direction of this axis is from metal 1 to metal 2, then the expression on the left-hand side of this equation is the amount of heat taken from the surface per unit time and area by thermal conduction. This heat loss is balanced by the evolution at the junction of an amount of heat given by the right-hand side of the equation. Thus the amount of heat generated per unit time and area is

$$j \Pi_{12}, \text{ where } \Pi_{12} = -T(\alpha_2 - \alpha_1). \quad (26.8)$$

The quantity  $\Pi_{12}$  is called the *Peltier coefficient*. Like the Thomson effect, the Peltier effect

is proportional to the first power of the current, and changes sign when the direction of the current is reversed. The Peltier coefficient is additive:  $\Pi_{13} = \Pi_{12} + \Pi_{23}$ , where the suffixes 1, 2, 3 refer to three different metals.

A comparison of formulae (26.7) and (26.8) shows that the Thomson and Peltier coefficients are related by

$$\rho_2 - \rho_1 = T \frac{d}{dT} \left( \frac{\Pi_{12}}{T} \right). \quad (26.9)$$

Next, let us consider an open circuit containing two junctions, the two end conductors being of the same metal (1 in Fig. 17). We suppose that the junctions  $b$  and  $c$  are at different temperatures  $T_1$  and  $T_2$ , while the temperature at each end ( $a$  and  $d$ ) is the same. Then there is a potential difference called a *thermoelectromotive force*, which we denote by  $\mathcal{E}_T$ , between the ends.

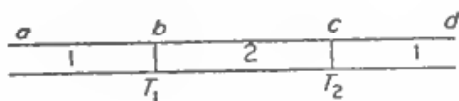


FIG. 17

To calculate this force, we put  $\mathbf{j} = 0$  in (26.1) and integrate the field  $\mathbf{E} = \alpha \text{grad } T$  along the circuit (taken to be the x-axis):

$$\mathcal{E}_T = \int_a^d \alpha \frac{dT}{dx} dx = \int_a^d \alpha dT.$$

The integrations from  $a$  to  $b$  and from  $c$  to  $d$  are over temperatures from  $T_2$  to  $T_1$  in metal 1, and that from  $b$  to  $c$  is over temperatures from  $T_1$  to  $T_2$  in metal 2. Thus

$$\mathcal{E}_T = \int_{T_1}^{T_2} (\alpha_2 - \alpha_1) dT. \quad (26.10)$$

Comparing this with (26.8), we see that the thermo-e.m.f. is related to the Peltier coefficient by

$$\mathcal{E}_T = - \int_{T_1}^{T_2} \frac{\Pi_{12}}{T} dT \quad (26.11)$$

Formulae (26.9) and (26.11) are called *Thomson's relations* (W. Thomson, 1854).

To conclude this section, we shall give the formulae for the current and heat flux in an anisotropic conductor. These are derived from the symmetry of the kinetic coefficients in the same way as formulae (26.1) and (26.4), and the results are

$$\left. \begin{aligned} E_i &= \sigma^{-1}_{ik} j_k + \alpha_{ik} \partial T / \partial x_k, \\ q_i - \phi j_i &= T \alpha_{ki} j_k - \kappa_{ik} \partial T / \partial x_k. \end{aligned} \right\} \quad (26.12)$$

Here  $\sigma^{-1}_{ik}$  is the tensor inverse to the conductivity tensor  $\sigma_{ik}$ , and the tensors  $\sigma_{ik}$  and  $\kappa_{ik}$  are symmetrical. The thermoelectric tensor  $\alpha_{ik}$ , however, is in general not symmetrical.

### §27. Thermogalvanomagnetic phenomena

There is a still greater variety of phenomena which occur when a current flows in the simultaneous presence of an electric field, a magnetic field, and a temperature gradient.

The discussion is entirely similar to that given in §26 for thermoelectric effects. It will be given here in tensor form, so as to be applicable to both isotropic and anisotropic conductors. We write the electric current density  $\mathbf{j}$  and the heat flux  $\mathbf{q}$  as

$$\left. \begin{aligned} j_i &= a_{ik} \frac{E_k}{T} + b_{ik} \frac{\partial}{\partial x_k} \left( \frac{1}{T} \right), \\ q_i - \phi j_i &= c_{ik} \frac{E_k}{T} + d_{ik} \frac{\partial}{\partial x_k} \left( \frac{1}{T} \right), \end{aligned} \right\} \quad (27.1)$$

where all the coefficients are functions of the magnetic field. The symmetry of the kinetic coefficients gives

$$\left. \begin{aligned} a_{ik}(\mathbf{H}) &= a_{ki}(-\mathbf{H}), & d_{ik}(\mathbf{H}) &= d_{ki}(-\mathbf{H}), \\ b_{ik}(\mathbf{H}) &= c_{ki}(-\mathbf{H}). \end{aligned} \right\} \quad (27.2)$$

Expressing  $\mathbf{E}$  and  $\mathbf{q} - \phi \mathbf{j}$  in terms of  $\mathbf{j}$  and  $\text{grad } T$  from (27.1), we have

$$\left. \begin{aligned} E_i &= \sigma^{-1}_{ik} j_k + \alpha_{ik} \partial T / \partial x_k, \\ q_i - \phi j_i &= \beta_{ik} j_k - \kappa_{ik} \partial T / \partial x_k, \end{aligned} \right\} \quad (27.3)$$

where the tensors  $\sigma^{-1}$ ,  $\alpha$ ,  $\beta$ ,  $\kappa$  are certain functions of the tensors  $a$ ,  $b$ ,  $c$ ,  $d$ , and have the following symmetry properties resulting from (27.2):

$$\left. \begin{aligned} \sigma^{-1}_{ik}(\mathbf{H}) &= \sigma^{-1}_{ki}(-\mathbf{H}), \\ \kappa_{ik}(\mathbf{H}) &= \kappa_{ki}(-\mathbf{H}), & \beta_{ik}(\mathbf{H}) &= T \alpha_{ki}(-\mathbf{H}). \end{aligned} \right\} \quad (27.4)$$

These are the required relations in their most general form. They generalize those found in §26 for the case where there is no magnetic field and in §22 for the case where there is no temperature gradient. It must be emphasized that in an anisotropic conductor the tensors  $\alpha_{ik}$  and  $\beta_{ik}$  are in general not symmetrical even when there is no magnetic field.

The tensors  $\sigma^{-1}$ ,  $\kappa$ , and  $\beta + T\alpha$  can be resolved into symmetrical and antisymmetrical parts (cf. §22). In a weak magnetic field, the symmetrical parts may be regarded as constants independent of  $\mathbf{H}$ , while the antisymmetric parts are linear in  $\mathbf{H}$ . For an isotropic conductor we have, to this accuracy,

$$\mathbf{E} = \mathbf{j} / \sigma + \alpha \text{grad } T + R \mathbf{H} \times \mathbf{j} + N \mathbf{H} \times \text{grad } T, \quad (27.5)$$

$$\mathbf{q} - \phi \mathbf{j} = \alpha T \mathbf{j} - \kappa \text{grad } T + N T \mathbf{H} \times \mathbf{j} + L \mathbf{H} \times \text{grad } T. \quad (27.6)$$

Here  $\sigma$  and  $\kappa$  are the ordinary coefficients of electrical and thermal conductivity,  $\alpha$  is the thermoelectric coefficient which appears in (26.1),  $R$  is the Hall coefficient, and  $N$  and  $L$  are new coefficients. The term  $N \mathbf{H} \times \text{grad } T$  may be regarded as representing the effect of the magnetic field on the thermo-e.m.f. (called the *Nernst effect*), and the term  $L \mathbf{H} \times \text{grad } T$  as representing the effect of this field on the thermal conduction (called the *Leduc-Righi effect*).

At a boundary between media, the normal components of the vectors  $\mathbf{j}$  and  $\mathbf{q}$  are



continuous, and therefore so is that of the vector  $-\kappa \mathbf{grad} T + \alpha T \mathbf{j} + NTH \times \mathbf{j} + LH \times \mathbf{grad} T$ . The term  $NTH \times \mathbf{j}$  gives the influence of the magnetic field on the Peltier effect (called the *Ettingshausen effect*).

The amount of heat evolved in the conductor per unit time and volume is  $Q = -\text{div } \mathbf{q}$ . Here we must substitute  $\mathbf{q}$  from (27.6) and replace  $-\mathbf{grad} \phi = \mathbf{E}$  in accordance with (27.5). If the conductor is homogeneous, then the quantities  $\alpha$ ,  $N$ ,  $L$ , etc. are functions of temperature alone, and so their gradients are proportional to  $\mathbf{grad} T$ . In the calculation we neglect all quantities of the second order in  $\mathbf{H}$ , and to the same approximation we can take  $\text{curl}(\mathbf{j}/\sigma) \cong \text{curl } \mathbf{E} = 0$ . We also note that the external field  $\mathbf{H}$  (arising from sources outside the conductor under consideration) is such that  $\text{curl } \mathbf{H} = 0$ .† Finally,  $\text{div } \mathbf{j} = 0$ , as for any steady current. The result is

$$Q = \frac{j^2}{\sigma} + \text{div}(\kappa \mathbf{grad} T) - T \mathbf{j} \cdot \mathbf{grad} \alpha + \frac{1}{\sigma T} \frac{d}{dT} (\sigma N T^2) (\mathbf{j} \times \mathbf{H}) \cdot \mathbf{grad} T.$$

The third term here gives the Thomson effect (26.7), and the last term gives the change in the Thomson effect resulting from the presence of the magnetic field.

## §28. Diffusion phenomena

The presence of diffusion causes certain phenomena in electrolyte solutions which do not occur in solid conductors. We shall assume, for simplicity, that the temperature is the same everywhere in the solution, and so consider only pure diffusion phenomena, uncomplicated by thermoelectric effects.

Instead of the pressure  $P$  and the concentration  $c$ , it is more convenient to take as independent variables the pressure and the chemical potential  $\zeta$ . We here define  $\zeta$  as the derivative of the thermodynamic potential of unit mass of the solution with respect to its concentration  $c$  (at constant  $P$  and  $T$ ); by the concentration we mean the ratio of the mass of electrolyte in a volume element to the total mass of fluid in the same volume.‡ It may be recalled that the constancy of the chemical potential is (like that of the pressure and the temperature) one of the conditions of thermodynamic equilibrium.

The definition of the electric field potential given in §26 has to be somewhat modified in this case, since the current is now carried by the ions of the dissolved electrolyte, and not by the conduction electrons. A suitable definition is (cf. the third footnote to §26)  $\phi = (\partial \Phi / \partial \rho)_c$ , where  $\Phi$  is the thermodynamic potential and  $\rho$  the sum of the ion charges in unit volume of the solution (after differentiating we put  $\rho = 0$ , of course, because the

† This neglects the weak effect on the evolution of heat resulting from the magnetic fields of the currents themselves.

‡ The chemical potentials are usually defined as  $\zeta_1 = \partial \Phi / \partial n_1$ ,  $\zeta_2 = \partial \Phi / \partial n_2$ , where  $\Phi$  is the thermodynamic potential of any mass of the solution, and  $n_1$ ,  $n_2$  the numbers of particles of solute and solvent in that mass of solution. If  $\Phi$  is the thermodynamic potential per unit mass, then the numbers  $n_1$  and  $n_2$  are related by  $n_1 m_1 + n_2 m_2 = 1$  (where  $m_1$ ,  $m_2$  are the masses of the two kinds of particle), and the concentration  $c = n_1 m_1$ . Hence we have

$$\zeta = \frac{\partial \Phi}{\partial c} = \frac{\partial \Phi}{\partial n_1} \frac{\partial n_1}{\partial c} + \frac{\partial \Phi}{\partial n_2} \frac{\partial n_2}{\partial c} = \frac{\zeta_1}{m_1} - \frac{\zeta_2}{m_2},$$

where  $\zeta$  is the chemical potential as here defined.



solution is electrically neutral). The derivative is taken at constant mass concentration, i.e. at a given sum of the masses of ions of both signs in unit volume.<sup>†</sup>

When a gradient of the chemical potential is present, a term proportional to it is added to the expression for the current density:

$$\mathbf{j} = \sigma(\mathbf{E} - \beta \mathbf{grad} \zeta), \quad (28.1)$$

in analogy with the added term in (26.2). We shall see below that, for a given gradient of the chemical potential (and of the temperature),  $\mathbf{j}$  must be independent of the pressure gradient, and so no term in  $\mathbf{grad} P$  appears in (28.1).<sup>‡</sup>

As well as the electric current, we have to consider the transport of the mass of the electrolyte which takes place at the same time. It must be borne in mind that the passage of a current through the solution may be accompanied by a macroscopic motion of the fluid. The mass flux density of the electrolyte resulting from this motion is  $\rho \mathbf{cv}$ , where  $\mathbf{v}$  is the velocity and  $\rho$  the density of the solution. The electrolyte is also transported by molecular diffusion. We denote the diffusion flux density by  $\mathbf{i}$ , so that the total flux density is  $\rho \mathbf{cv} + \mathbf{i}$ . The irreversible processes of diffusion cause a further increase in entropy; the rate of change of the total entropy is§

$$\frac{d\mathcal{S}}{dt} = \int \frac{\mathbf{E} \cdot \mathbf{j}}{T} dV - \int \frac{\mathbf{i} \cdot \mathbf{grad} \zeta}{T} dV. \quad (28.2)$$

Like the electric current density, the diffusion flux may be written as a linear combination of  $\mathbf{E}$  and  $\mathbf{grad} \zeta$ , or of  $\mathbf{j}$  and  $\mathbf{grad} \zeta$ . Using the symmetry of the kinetic coefficients, we can relate one of the coefficients in this combination to the coefficient  $\beta$  in (28.1), in exactly the same way as we did for  $\mathbf{j}$  and  $\mathbf{q} - \phi \mathbf{j}$  in §26. The result is

$$\mathbf{i} = -\frac{\rho D}{(\partial \zeta / \partial c)_{P,T}} \mathbf{grad} \zeta + \beta \mathbf{j}. \quad (28.3)$$

The coefficient of  $\mathbf{grad} \zeta$  is here expressed in terms of the ordinary diffusion coefficient,  $D$  being the density of matter. For  $\mathbf{j} = 0$  and constant pressure and temperature we have  $\mathbf{i} = -\rho D \mathbf{grad} c$ .

The inadmissibility in (28.1) and (28.3) of terms proportional to the pressure gradient follows, as in §26, from the law of the increase of entropy: such terms would make the derivative of the total entropy (28.2) a quantity of variable sign.

Formulae (28.1) and (28.3) give all the diffusion phenomena in electrolytes, but we shall not pause here to examine them more closely.

## PROBLEM

Two parallel plates of a metal  $A$  are immersed in a solution of an electrolyte  $AX$ . Find the current density as a function of the potential difference applied between the plates.

<sup>†</sup> In a strong electrolyte, the solute is completely dissociated, and so the mass concentration may be written as  $c = m_+ n_+ + m_- n_-$ , where  $m_+$  and  $m_-$  are the cation and anion masses,  $n_+$  and  $n_-$  their number densities. With the above definition of the potential,  $\phi = 0$  corresponds to  $\zeta_+ / m_+ = \zeta_- / m_-$  for the cation and anion chemical potentials, which are also related by  $\zeta_+ + \zeta_- = \zeta_1$ .

<sup>‡</sup> It should be emphasized, however, that, for a given concentration gradient  $\mathbf{j}$  does depend on the pressure gradient:

$$\mathbf{grad} \zeta = (\partial \zeta / \partial c)_{P,T} \mathbf{grad} c + (\partial \zeta / \partial P)_{c,T} \mathbf{grad} P.$$

§ The derivation of the second term is given in *FM*, §57.

SOLUTION. When the current passes, metal is dissolved from one plate and deposited on the other. The solvent (water) remains at rest, and a mass flux of metal of density†  $\rho v = jm/e$  occurs in the solution, where  $j$  is the electric current density, and  $m$  and  $e$  are the mass and charge of an ion  $A^+$ . This flux is also given by  $i + \rho v c$ , where  $i$  is as shown in (28.3); assuming the pressure constant throughout the liquid,‡ we have

$$\rho D \frac{dc}{dx} = \left[ \beta - \frac{m}{e}(1-c) \right] j, \quad (1)$$

where  $x$  is the coordinate in the direction of a line joining the electrodes. Since  $j = \text{constant}$  in the solution, this gives

$$jl = \int_{c_1}^{c_2} \frac{\rho D dc}{\beta - m(1-c)/e}, \quad (2)$$

where  $c_1, c_2$  are the concentrations at the surfaces of the plates, and  $l$  is the distance between them.

The potential difference  $\mathcal{E}$  between the plates is most simply found from the total amount of energy  $Q$  dissipated per unit time and unit area of the plates, which must equal  $j\mathcal{E}$ . By (28.1), (28.2) we have

$$Q = T \frac{d\mathcal{S}}{dt} = \int \left\{ \frac{j^2}{\sigma} + \rho D \frac{\partial \zeta}{\partial c} \left( \frac{dc}{dx} \right)^2 \right\} dx = j\mathcal{E},$$

and therefore, using (1),

$$\mathcal{E} = \int_{c_1}^{c_2} \frac{\rho D dc}{\sigma(\beta - m(1-c)/e)} + \int_{c_1}^{c_2} \frac{\partial \zeta}{\partial c} \left[ \beta - \frac{m}{e}(1-c) \right] dc. \quad (3)$$

Formulae (2) and (3) implicitly solve the problem.

If the current  $j$  is small, the concentration difference  $c_2 - c_1$  is also small. Replacing the integrals by  $c_2 - c_1$  times the integrands, we find the effective specific resistance of the solution:

$$\frac{\mathcal{E}}{lj} = \frac{1}{\sigma} + \frac{1}{\rho D} \frac{\partial \zeta}{\partial c} \left[ \beta - \frac{m}{e}(1-c) \right]^2.$$

The first term in (3) gives the potential drop ( $\int (j/\sigma) dx$ ) due to the passage of the current. The second term is the e.m.f. due to the concentration gradient in the solution (in a certain sense analogous to the thermo-e.m.f.). This latter expression is independent of the conditions of the particular one-dimensional problem considered, and is the general expression for the e.m.f. of a "concentration cell".

† It may be recalled that the hydrodynamic velocity  $v$  in a solution is defined so that  $\rho v$  is the momentum of unit volume of the liquid; see *FM*, §57. Hence the fact that in this case only the dissolved metal is moving (relative to the electrodes) does not affect the calculation of  $\rho v$ .

‡ The change in pressure due to the motion of the liquid gives only terms of a higher order of smallness.

## CHAPTER IV

# STATIC MAGNETIC FIELD

### §29. Static magnetic field

A STATIC magnetic field in matter satisfies two of Maxwell's equations, obtained by averaging the microscopic equations

$$\operatorname{div} \mathbf{h} = 0, \quad \operatorname{curl} \mathbf{h} = \frac{1}{c} \frac{\partial \mathbf{e}}{\partial t} + \frac{4\pi}{c} \rho \mathbf{v}. \quad (29.1)$$

The mean magnetic field is usually called the *magnetic induction* and denoted by  $\mathbf{B}$ :

$$\overline{\mathbf{h}} = \mathbf{B}. \quad (29.2)$$

Hence the result of averaging the first equation (29.1) is

$$\operatorname{div} \mathbf{B} = 0. \quad (29.3)$$

In the second equation, the time derivative gives zero on averaging, since the mean field is supposed constant, and so we have

$$\operatorname{curl} \mathbf{B} = (4\pi/c) \overline{\rho \mathbf{v}}. \quad (29.4)$$

The mean value of the microscopic current density is in general not zero in either conductors or dielectrics. The only difference between these two classes is that in dielectrics we always have

$$\int \overline{\rho \mathbf{v}} \cdot d\mathbf{f} = 0, \quad (29.5)$$

where the integral is taken over the area of any cross-section of the body; in conductors, this integral need not be zero. Let us suppose to begin with that there is no net current in the body if it is a conductor, i.e. that (29.5) holds.

The vanishing of the integral in (29.5) for every cross-section of the body means that the vector  $\overline{\rho \mathbf{v}}$  can be written as the curl of another vector, usually denoted by  $c\mathbf{M}$ :

$$\overline{\rho \mathbf{v}} = c \operatorname{curl} \mathbf{M}, \quad (29.6)$$

where  $\mathbf{M}$  is zero outside the body; compare the similar discussion in §6. For, integrating over a surface bounded by a curve which encloses the body and nowhere enters it, we have  $\int \overline{\rho \mathbf{v}} \cdot d\mathbf{f} = c \int \operatorname{curl} \mathbf{M} \cdot d\mathbf{f} = c \oint \mathbf{M} \cdot d\mathbf{l} = 0$ . The vector  $\mathbf{M}$  is called the *magnetization* of the body. Substituting it in (29.4), we find

$$\operatorname{curl} \mathbf{H} = 0, \quad (29.7)$$

where the vector  $\mathbf{H}$  and the magnetic induction  $\mathbf{B}$  are related by

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}, \quad (29.8)$$

which is analogous to the relation between the electric field  $\mathbf{E}$  and induction  $\mathbf{D}$ . Although  $\mathbf{H}$  is, by analogy with  $\mathbf{E}$ , usually called the *magnetic field*, it must be remembered that the true mean field is really  $\mathbf{B}$  and not  $\mathbf{H}$ .

To see the physical significance of the quantity  $\mathbf{M}$ , let us consider the total magnetic moment due to all the charged particles moving in the body. By the definition of the magnetic moment (see *Fields*, §44), this is†

$$\int \mathbf{r} \times \bar{\rho} \mathbf{v} dV / 2c = \frac{1}{2} \int \mathbf{r} \times \text{curl } \mathbf{M} dV.$$

Since  $\rho \mathbf{v} \equiv 0$  outside the body, the integral can be taken over any volume which includes the body. We transform the integral as follows:

$$\int \mathbf{r} \times \text{curl } \mathbf{M} dV = - \oint \mathbf{r} \times (\mathbf{M} \times d\mathbf{f}) - \int (\mathbf{M} \times \text{grad}) \times \mathbf{r} dV.$$

The integral over the surface outside the body is zero. In the second term we have  $(\mathbf{M} \times \text{grad}) \times \mathbf{r} = -\mathbf{M} \text{div } \mathbf{r} + \mathbf{M} = -2\mathbf{M}$ . Thus we obtain

$$\frac{1}{2c} \int \mathbf{r} \times \bar{\rho} \mathbf{v} dV = \int \mathbf{M} dV. \quad (29.9)$$

We see that the magnetization vector is the magnetic moment per unit volume.‡

The equations (29.3) and (29.7) must be supplemented by a relation between  $\mathbf{H}$  and  $\mathbf{B}$  in order to complete the system of equations. For example, in non-ferromagnetic bodies in fairly weak magnetic fields,  $\mathbf{B}$  and  $\mathbf{H}$  are linearly related. In isotropic bodies, this linear relation becomes a simple proportionality:

$$\mathbf{B} = \mu \mathbf{H}. \quad (29.10)$$

The coefficient  $\mu$  is called the *magnetic permeability*. We also have  $\mathbf{M} = \chi \mathbf{H}$ , where the coefficient

$$\chi = (\mu - 1)/4\pi \quad (29.11)$$

is called the *magnetic susceptibility*.

Unlike the permittivity  $\epsilon$ , which always exceeds unity, the magnetic permeability may be either greater or less than unity. (It is, however, always positive, as we shall prove in §31. The reason for the differing behaviour of  $\mu$  and  $\epsilon$  is discussed in §32.) The magnetic susceptibility  $\chi$  may correspondingly be either positive or negative.

Another, quantitative, difference is that the magnetic susceptibility of the great majority of bodies is very small in comparison with the dielectric susceptibility. This difference arises because the magnetization of a (non-ferromagnetic) body is a relativistic effect, of order  $v^2/c^2$ , where  $v$  is the velocity of the electrons in the atoms.§

In anisotropic bodies (crystals), the simple proportionality (29.10) is replaced by the linear relations

$$B_i = \mu_{ik} H_k. \quad (29.12)$$

† For clarity, it should be emphasized that  $\mathbf{r}$  here is a variable coordinate of integration, not the position vector of a microscopic particle; it therefore does not come under the averaging sign.

‡ The quantity  $\mathbf{M}$  is completely determined only when this relation is established. The relation (29.6) inside the body, and  $\mathbf{M} = 0$  outside it, do not uniquely define  $\mathbf{M}$ : the gradient of any scalar could be added to  $\mathbf{M}$  inside the body without affecting (29.6) (cf. the similar remark in the first footnote to §6).

§ The ratio  $v/c$  appears with  $\mathbf{H}$  in the Hamiltonian of the interaction of the body with the magnetic field, and again in the magnetic moments of the atoms or molecules.

The magnetic permeability tensor  $\mu_{ik}$  is symmetrical; this follows from the thermodynamic relations to be derived in §31, in exactly the same way as for  $\epsilon_{ik}$  (§13).

From the equations  $\text{div } \mathbf{B} = 0$ ,  $\text{curl } \mathbf{H} = 0$  it follows (cf. §6) that at a boundary between two different media we must have

$$B_{1n} = B_{2n}, \quad H_{1t} = H_{2t}. \quad (29.13)$$

This system of equations and boundary conditions is formally identical with those for the electrostatic field in a dielectric in the absence of free charges, differing only in that  $\mathbf{E}$  and  $\mathbf{D}$  are replaced by  $\mathbf{H}$  and  $\mathbf{B}$  respectively. Since  $\text{curl } \mathbf{H} = 0$ , we can put  $\mathbf{H} = -\text{grad } \psi$ ; the equations for the potential  $\psi$  are the same as those for the electrostatic potential. Thus the solutions of the various problems of electrostatics discussed in Chapter II can be immediately applied to problems with a static magnetic field. In particular, the formulae derived in §8 for a dielectric ellipsoid in a uniform electric field hold also, with appropriate substitutions, for a magnetic ellipsoid in a uniform magnetic field. For example, the magnetic field  $\mathbf{H}^{(i)}$  and induction  $\mathbf{B}^{(i)}$  inside the ellipsoid are related to the external field  $\mathfrak{H}$  by

$$H_i^{(i)} + n_{ik}(B_k^{(i)} - H_k^{(i)}) = \mathfrak{H}_i, \quad (29.14)$$

where  $n_{ik}$  is the demagnetizing factor tensor. This relation is valid, whatever the relation between  $\mathbf{B}$  and  $\mathbf{H}$ .

The tangential component of the magnetic induction, unlike its normal component, is discontinuous at a surface separating two media. The magnitude of the discontinuity can be related to the current density on the surface. To do this, we integrate both sides of equation (29.4) over a small interval  $\Delta l$  crossing the surface along the normal. We then let  $\Delta l$  tend to zero; the integral  $\int \rho \bar{\mathbf{v}} dl$  may tend to some finite limit. The quantity

$$\mathbf{g} = \int \rho \bar{\mathbf{v}} dl \quad (29.15)$$

may be called the *surface current density*; it gives the charge passing per unit time across unit length of a line in the surface. We take the direction of  $\mathbf{g}$  at a given point on the surface as the  $y$ -axis, and the direction of the normal from medium 1 to medium 2 as the  $x$ -axis. Then the integration of equation (29.4) gives

$$\int \left( \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) dx = \frac{4\pi}{c} g_y = \frac{4\pi}{c} g.$$

Since  $B_x$  is continuous, the derivative  $\partial B_x / \partial z$  is finite, and so its integral tends to zero with  $\Delta l$ . The integral of  $\partial B_z / \partial x$  gives the difference in the values of  $B_z$  on the two sides of the surface. Thus  $B_{2z} - B_{1z} = -4\pi g/c$ . This can be written in vector form:

$$4\pi \mathbf{g}/c = \mathbf{n} \times (\mathbf{B}_2 - \mathbf{B}_1) = 4\pi \mathbf{n} \times (\mathbf{M}_2 - \mathbf{M}_1), \quad (29.16)$$

where  $\mathbf{n}$  is a unit vector along the normal into region 2; the last member of (29.16) is obtained by using the continuity of the tangential component of  $\mathbf{H}$ .

### §30. The magnetic field of a steady current

If a conductor carries a non-zero total current, the mean current density in it can be written as  $\bar{\rho} \bar{\mathbf{v}} = c \text{curl } \mathbf{M} + \mathbf{j}$ . The first term, resulting from the magnetization of the medium, makes no contribution to the total current, so that the net charge transfer through

a cross-section of the body is given by the integral  $\int \mathbf{j} \cdot d\mathbf{f}$  of the second term. The quantity  $\mathbf{j}$  is called the *conduction current density*.† The statements made in §21 apply to this current; in particular, the energy dissipated per unit time and volume is  $\mathbf{E} \cdot \mathbf{j}$ .

The distribution of the current  $\mathbf{j}$  over the volume of the conductor is given by the equations of §21, which do not involve the magnetic field due to  $\mathbf{j}$  itself, if we neglect the effect of this field on the conductivity of the body. Hence the magnetic field of the currents must be determined for a given current distribution. The equations satisfied by this field differ from those in §29 by the presence of a term  $4\pi\mathbf{j}/c$  on the right-hand side of (29.7):

$$\operatorname{div} \mathbf{B} = 0, \quad (30.1)$$

$$\operatorname{curl} \mathbf{H} = 4\pi\mathbf{j}/c. \quad (30.2)$$

The conduction current density  $\mathbf{j}$ , which is proportional to the electric field, does not become infinite, and in particular is finite on a surface separating two media. Hence the term on the right of (30.2) does not affect the boundary condition that the tangential component of  $\mathbf{H}$  be continuous.

To solve equations (30.1), (30.2), it is convenient to use the *vector potential*  $\mathbf{A}$ , defined by

$$\mathbf{B} = \operatorname{curl} \mathbf{A}, \quad (30.3)$$

so that equation (30.1) is satisfied identically. Equation (30.3) does not uniquely define the vector potential, to which the gradient of any scalar may be added without affecting (30.3). For this reason we can impose on  $\mathbf{A}$  a further condition, which we take to be

$$\operatorname{div} \mathbf{A} = 0. \quad (30.4)$$

The equation for  $\mathbf{A}$  is obtained by substituting (30.3) in (30.2). If the linear relation  $\mathbf{B} = \mu\mathbf{H}$  holds we have

$$\operatorname{curl} \left( \frac{1}{\mu} \operatorname{curl} \mathbf{A} \right) = 4\pi\mathbf{j}/c. \quad (30.5)$$

In this form the equation is valid for any medium, homogeneous or not.

In a homogeneous medium,  $\mu = \text{constant}$ , and since

$$\operatorname{curl} \operatorname{curl} \mathbf{A} = \operatorname{grad} \operatorname{div} \mathbf{A} - \Delta \mathbf{A} = -\Delta \mathbf{A}$$

we find from (30.5)

$$\Delta \mathbf{A} = -4\pi\mu\mathbf{j}/c. \quad (30.6)$$

If we have two or more adjoining media with different magnetic permeability  $\mu$ , the general equation (30.5) has the form (30.6) in each homogeneous medium, while at the interfaces the tangential component of the vector  $(1/\mu) \operatorname{curl} \mathbf{A}$  must be continuous. Moreover, the tangential component of  $\mathbf{A}$  itself must be continuous, since a discontinuity would mean that the induction  $\mathbf{B}$  was infinite at the boundary.

The field equations are simpler in the two-dimensional problem of finding the magnetic field in a medium infinite and homogeneous in one direction (which we take as the  $z$ -direction), the currents which produce the field being everywhere in that direction, with the

† The quantity  $c \operatorname{curl} \mathbf{M}$  is sometimes called the *molecular current density*. This name, however, is not in complete accordance with the actual physical picture of motion of charges in a conductor. For example, in a metal the conduction electrons, as well as those moving in the atoms, contribute to the magnetization.

current density  $j_z = j$  depending only on  $x$  and  $y$ . We make the plausible assumption (to be confirmed by the result) that the vector potential of such a field is also in the  $z$ -direction:  $A_z = A(x, y)$ . The condition (30.4) is then satisfied identically; the magnetic field is everywhere parallel to the  $xy$ -plane. We denote by  $\mathbf{k}$  a unit vector in the  $z$ -direction; then

$$\mathbf{curl} \mathbf{A} = \mathbf{curl} A \mathbf{k} = \mathbf{grad} A \times \mathbf{k},$$

$$\mathbf{curl} \left( \frac{1}{\mu} \mathbf{curl} \mathbf{A} \right) = \mathbf{curl} \left( \frac{\mathbf{grad} A}{\mu} \times \mathbf{k} \right) = -\mathbf{k} \operatorname{div} \frac{\mathbf{grad} A}{\mu}.$$

Hence equation (30.5) becomes

$$\operatorname{div} \frac{\mathbf{grad} A}{\mu} = -\frac{4\pi}{c} j(x, y), \quad (30.7)$$

i.e. we in fact obtain one equation for the one scalar quantity  $A(x, y)$ . For a piecewise homogeneous medium, (30.7) becomes

$$\Delta A = -4\pi\mu j(x, y)/c, \quad (30.8)$$

with the boundary condition that  $A$  and  $(1/\mu)\partial A/\partial n$  be continuous at an interface.†

The magnetic field is easily found if the current distribution is symmetrical about the  $z$ -axis:  $j_z = j(r)$  (where  $r$  is the distance from that axis). In this case the lines of magnetic force are evidently the circles  $r = \text{constant}$ . The magnitude of the field is found at once from the formula

$$\oint \mathbf{H} \cdot d\mathbf{l} = \frac{4\pi}{c} \int \mathbf{j} \cdot d\mathbf{f}, \quad (30.9)$$

which is the integral form of (30.2). Thus

$$H(r) = 2J(r)/cr, \quad (30.10)$$

where  $J(r)$  is the total current within the radius  $r$ .

The reduction of the vector equation (30.5) to a single scalar equation is possible also if the current distribution is axially symmetrical and has in cylindrical polar coordinates  $r, \phi, z$  the form  $j_r = j_z = 0, j_\phi = j(r, z)$ . We seek the vector potential in the form  $A_r = A_z = 0, A_\phi = A(r, z)$ . The components of the magnetic induction  $\mathbf{B} = \mathbf{curl} \mathbf{A}$  are  $B_r = -\partial A/\partial z, B_z = (1/r)\partial(rA)/\partial r, B_\phi = 0$ , and the  $\phi$ -component of equation (30.2) gives

$$\frac{\partial}{\partial z} \left( \frac{1}{\mu} \frac{\partial A}{\partial z} \right) + \frac{\partial}{\partial r} \left\{ \frac{1}{\mu r} \frac{\partial}{\partial r} (rA) \right\} = -\frac{4\pi}{c} j(r, z). \quad (30.11)$$

The equations for the magnetic field of the currents can be solved in a general form in the important case where the magnetic properties of the medium may be neglected, i.e. where we can put  $\mu = 1$ . The vector potential then satisfies in all space the equation

† It should be noticed that the two-dimensional problem with a static magnetic field is equivalent to the two-dimensional electrostatic problem of determining the electric field due to extraneous charges with density  $\rho_{ex}(x, y)$  in a dielectric medium. The equation to be solved in the latter problem is  $\operatorname{div}(\epsilon \mathbf{grad} \phi) = -4\pi\rho_{ex}$ , where  $\phi$  is the field potential; this differs from (30.7) only in that  $A, j/c$  and  $\mu$  are replaced by  $\phi, \rho_{ex}$  and  $1/\epsilon$  respectively. The boundary conditions on  $A$  and  $\phi$  are the same. A difference occurs, however, on passing to  $\mathbf{E}$  and  $\mathbf{B}$  from  $\phi$  and  $A$  respectively. The vectors  $\mathbf{E} = -\mathbf{grad} \phi$  and  $\mathbf{B} = \mathbf{curl} \mathbf{A}$  are the same in magnitude but in perpendicular directions at any given point.

$\triangle \mathbf{A} = -4\pi \mathbf{j}/c$  with no conditions at the interfaces between different media (including the surface of the conductor on which the current flows). The solution of this equation which vanishes at infinity is

$$\mathbf{A} = \frac{1}{c} \int \frac{\mathbf{j}}{R} dV, \quad (30.12)$$

where  $R$  is the distance from the volume element  $dV$  to the point at which  $\mathbf{A}$  is to be calculated (see *Fields*, §43). In taking the curl of this equation, we must remember that the integrand  $\mathbf{j}/R$  is to be differentiated with respect to the coordinates of this point, of which  $\mathbf{j}$  is independent, so that

$$\text{curl}(\mathbf{j}/R) = \text{grad}(1/R) \times \mathbf{j} = -\mathbf{R} \times \mathbf{j}/R^3,$$

where the radius vector  $\mathbf{R}$  is from  $dV$  to the point under consideration. Thus

$$\mathbf{B} = \mathbf{H} = \frac{1}{c} \int \frac{\mathbf{j} \times \mathbf{R}}{R^3} dV. \quad (30.13)$$

If the conductor on which the current flows is sufficiently thin (a thin wire), and if we are interested only in the field in the surrounding space, the thickness of the wire may be neglected. In what follows we shall often discuss such *linear currents*. The integration over the volume of the conductor is then replaced by an integration along its length: the formulae for linear currents are obtained from those for volume currents by making the substitution  $\mathbf{j} dV \rightarrow J d\mathbf{l}$ , where  $J$  is the total current in the conductor. For example, from formulae (30.12) and (30.13) we have

$$\mathbf{A} = \frac{J}{c} \oint \frac{d\mathbf{l}}{R}, \quad \mathbf{H} = \frac{J}{c} \oint \frac{d\mathbf{l} \times \mathbf{R}}{R^3}. \quad (30.14)$$

The latter formula is *Biot and Savart's law*.

This simple formula for the magnetic field of a linear current does not depend on the assumption that  $\mu = 1$ . Since we neglect the thickness of the conductor, no boundary conditions at its surface need be applied, and the magnetic properties of the conducting material are of no importance (it may even be ferromagnetic). The solution of equation (30.6) for the field in the medium surrounding the conductor is therefore

$$\mathbf{A} = \frac{\mu J}{c} \oint \frac{d\mathbf{l}}{R}, \quad \mathbf{B} = \frac{\mu J}{c} \oint \frac{d\mathbf{l} \times \mathbf{R}}{R^3}, \quad (30.15)$$

whatever the magnetic susceptibility of that medium. Thus the presence of the medium simply changes the magnetic induction by a factor  $\mu$ . The field  $\mathbf{H} = \mathbf{B}/\mu$  is unchanged.

The problem of determining the magnetic field of linear currents can also be solved as a problem of potential theory. Since we neglect the volume of conductors, we are in fact determining the field in a region containing no currents except along certain line singularities. In the absence of currents, a static magnetic field has a scalar potential, which in a homogeneous medium satisfies Laplace's equation. There is, however, an important difference between the magnetic field potential and the electrostatic potential: the latter is always a one-valued function, because  $\text{curl} \mathbf{E} = 0$  in all space (including charged regions) and so the change in the potential in going round any closed contour (i.e. the circulation of  $\mathbf{E}$  round that contour) is zero. The circulation of the magnetic field round a contour enclosing a linear current is not zero, but  $4\pi J/c$ . Hence the potential changes by this



amount on each passage round a contour enclosing a linear current, i.e. it is a many-valued function.

If the currents lie in a finite region of space (and  $\mu = 1$  everywhere), the vector potential of the magnetic field at a great distance from the conductors is

$$\mathbf{A} = \mathcal{M} \times \mathbf{R} / R^3, \quad (30.16)$$

where

$$\mathcal{M} = \int \mathbf{r} \times \mathbf{j} dV / 2c \quad (30.17)$$

is the total magnetic moment of the system.†

For a linear current, this becomes

$$\mathcal{M} = J \oint \mathbf{r} \times d\mathbf{l} / 2c,$$

and can be transformed into an integral over a surface bounded by the line of the current. The product  $d\mathbf{f} = \frac{1}{2} \mathbf{r} \times d\mathbf{l}$  is equal in magnitude to the area of the triangular surface element formed by the vectors  $\mathbf{r}$  and  $d\mathbf{l}$ . The vector  $\oint d\mathbf{f}$  is independent of the particular surface (bounded by the current) over which it is taken. Thus the magnetic moment of a closed linear current is

$$\mathcal{M} = J \oint d\mathbf{f} / c. \quad (30.18)$$

In particular, for a plane closed linear current the magnetic moment is simply  $JS/c$ , where  $S$  is the area of the plane enclosed by the current.

To conclude this section, we may briefly discuss the energy flux in a conductor. The energy dissipated as Joule heat in the conductor is derived from the energy of the electromagnetic field. In a steady state, the equation of continuity which expresses the law of conservation of energy is

$$-\operatorname{div} \mathbf{S} = \mathbf{j} \cdot \mathbf{E}, \quad (30.19)$$

where  $\mathbf{S}$  is the energy flux density, given in a conductor by

$$\mathbf{S} = c \mathbf{E} \times \mathbf{H} / 4\pi, \quad (30.20)$$

which is formally the same as the expression for the Poynting vector for the field in a vacuum. This is easily verified directly by calculating  $\operatorname{div} \mathbf{S}$  from the equations  $\operatorname{curl} \mathbf{E} = 0$  and (30.2), when we obtain (30.19).

Formula (30.20) also follows independently from the obvious condition that the normal component of  $\mathbf{S}$  must be continuous at the surface of a conductor, if we use the continuity of  $\mathbf{E}_t$  and  $\mathbf{H}_t$  and the validity of (30.20) in the vacuum outside the body.

### PROBLEMS‡

PROBLEM 1. Determine the scalar potential of the magnetic field of a closed linear current.

SOLUTION. We transform the curvilinear integral into one over a surface bounded by the curve, obtaining

$$\begin{aligned} \mathbf{A} &= \frac{J}{c} \oint \frac{d\mathbf{l}}{R} = \frac{J}{c} \int d\mathbf{f} \times \operatorname{grad} \frac{1}{R}, \\ \mathbf{B} &= \operatorname{curl} \mathbf{A} = -\frac{J}{c} \int (d\mathbf{f} \cdot \operatorname{grad}) \operatorname{grad} \frac{1}{R} \end{aligned}$$

† See *Fields*, §44. In the derivation there given, we use explicitly the idea of a current as the result of the motion of individual charged particles. Such a derivation is, of course, quite general, but formula (30.16) can also be obtained by macroscopic arguments (see Problem 4).

‡ In Problems 1–4,  $\mu = 1$ .

(where we have used the fact that  $\Delta(1/R) = 0$ ). Since  $\mathbf{B} = -\text{grad } \psi$ , we have for the scalar potential

$$\psi = \frac{J}{c} \int \mathbf{df} \cdot \text{grad} \frac{1}{R} = -\frac{J}{c} \int \frac{\mathbf{df} \cdot \mathbf{R}}{R^3}.$$

The integral is, geometrically, the solid angle  $\Omega$  subtended by the closed contour at the point considered. The above-mentioned many-valuedness of the potential is seen from the fact that, as this point describes a closed path embracing the wire, the angle  $\Omega$  changes suddenly from  $2\pi$  to  $-2\pi$ .

**PROBLEM 2.** Find the magnetic field of a linear current flowing in a circle with radius  $a$ .

**SOLUTION.** We take the origin of cylindrical polar coordinates  $r, \phi, z$  at the centre of the circle, with the angle  $\phi$  measured from the plane which passes through the  $z$ -axis and the point at which the field is calculated. The vector potential has only one component,  $A_\phi = A(r, z)$ , and by formula (30.14) we have

$$\begin{aligned} A_\phi &= \frac{J}{c} \oint \frac{\cos \phi \, dl}{R} \\ &= \frac{2J}{c} \int_0^\pi \frac{a \cos \phi \, d\phi}{\sqrt{(a^2 + r^2 + z^2 - 2ar \cos \phi)}} \end{aligned}$$

Putting  $\theta = \frac{1}{2}(\phi - \pi)$ , we find

$$A_\phi = \frac{4J}{ck} \sqrt{\frac{a}{r}} \left[ \left(1 - \frac{1}{2}k^2\right)K - E \right],$$

where  $k^2 = 4ar/[(a+r)^2 + z^2]$ , and  $K$  and  $E$  are complete elliptic integrals of the first and second kinds:

$$K = \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{(1 - k^2 \sin^2 \theta)}}, \quad E = \int_0^{\frac{1}{2}\pi} \sqrt{(1 - k^2 \sin^2 \theta)} \, d\theta.$$

The components of the induction are

$$\begin{aligned} B_\phi &= 0, \\ B_r &= -\frac{\partial A_\phi}{\partial z} = \frac{J}{c} \frac{2z}{r\sqrt{[(a+r)^2 + z^2]}} \left[ -K + \frac{a^2 + r^2 + z^2}{(a-r)^2 + z^2} E \right], \\ B_z &= \frac{1}{r} \frac{\partial}{\partial r} (rA_\phi) = \frac{J}{c} \frac{2}{\sqrt{[(a+r)^2 + z^2]}} \left[ K + \frac{a^2 - r^2 - z^2}{(a-r)^2 + z^2} E \right]. \end{aligned}$$

Here we have used the easily verified formulae

$$\frac{\partial K}{\partial k} = \frac{E}{k(1-k^2)} - \frac{K}{k}, \quad \frac{\partial E}{\partial k} = \frac{E-K}{k}.$$

On the axis ( $r = 0$ ) we have  $B_r = 0$ ,  $B_z = 2\pi a^2 J/c(a^2 + z^2)^{3/2}$ , as can also be found by a straightforward calculation.

**PROBLEM 3.** Determine the magnetic field in a cylindrical hole in a cylindrical conductor of infinite length carrying a current uniformly distributed over its cross-section (Fig. 18).

**SOLUTION.** If there were no hole, the field in the cylinder would be given by  $H'_x = -2\pi jy/c$ ,  $H'_y = 2\pi jx/c$ . The dimensions and axes are as shown in Fig. 18. If a current with density  $-j$  were to flow in the inner cylinder, it would produce a field  $H''_x = 2\pi jy'/c$ ,  $H''_y = -2\pi jx'/c$ . The required field in the hole is obtained by superposing these two fields. Since  $x - x' = OO' = h$ , and  $y = y'$ , we have  $H_x = 0$ ,  $H_y = 2\pi jh/c = 2hJ/(b^2 - a^2)c$ , i.e. a uniform field in the  $y$ -direction.

**PROBLEM 4.** Derive from (30.12) the formula (30.16) for the vector potential of the field far from the currents.

**SOLUTION.** We write  $\mathbf{R} = \mathbf{R}_0 - \mathbf{r}$ , where  $\mathbf{R}_0$  and  $\mathbf{r}$  are the radius vectors from the origin (situated somewhere among the currents) to the point considered and to the volume element  $dV$  respectively. Expanding the integrand

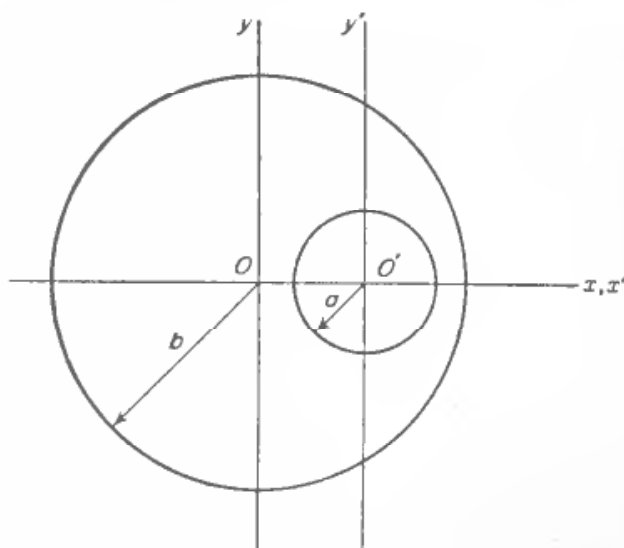


FIG. 18

in powers of  $r$  and using the fact that  $\int \mathbf{j} dV \equiv 0$ , we have  $A_i \cong (R_k/cR^3) \int x_k j_i dV$ . The suffix 0 to  $R$  is omitted. Integrating by parts the identity  $\int x_i x_k \operatorname{div} \mathbf{j} dV = 0$  gives  $\int (j_i x_k + j_k x_i) dV = 0$ . Hence we can write

$$A_i = (R_k/2cR^3) \int (x_k j_i - x_i j_k) dV,$$

which agrees with (30.16).

**PROBLEM 5.** Determine the magnetic field produced by a linear current in a magnetically anisotropic medium (A. S. Viglin, 1954).

**SOLUTION.** In the anisotropic medium surrounding the conductor we have

$$\operatorname{div} \mathbf{B} = \mu_{ik} \partial H_k / \partial x_i = 0, \quad (1)$$

where  $\mu_{ik}$  is the magnetic permeability tensor of the medium. Instead of introducing the vector potential by  $\mathbf{B} = \operatorname{curl} \mathbf{A}$ , we use another vector  $\mathbf{C}$  defined by

$$H_i = e_{ikl} \mu_{km} \partial C_l / \partial x_m, \quad (2)$$

where  $e_{ikl}$  is the antisymmetrical unit tensor. Then equation (1) is again satisfied identically. We can also impose on the vector  $\mathbf{C}$  thus defined the condition

$$\operatorname{div} \mathbf{C} \equiv \partial C_l / \partial x_l = 0. \quad (3)$$

Substituting (2) in  $\operatorname{curl} \mathbf{H} = 4\pi \mathbf{j}/c$ , we obtain  $e_{ikl} \partial H_l / \partial x_k = -\mu_{kp} \partial^2 C_l / \partial x_k \partial x_p = 4\pi j_i/c$  (using the condition (3) and the fact that  $e_{ikl} e_{lmn} = \delta_{im} \delta_{kn} - \delta_{in} \delta_{km}$ ). The equation thus obtained for  $\mathbf{C}$  is the same in form as that for the electric field potential resulting from charges in an anisotropic medium (§13, Problem 2). The solution is

$$\mathbf{C} = \frac{1}{c} \int \frac{\mathbf{j} dV}{\sqrt{(|\mu| \mu^{-1}_{ik} R_i R_k)}},$$

where  $|\mu|$  is the determinant of the tensor  $\mu_{ik}$ , and  $\mathbf{R}$  the radius vector from the point considered to  $dV$ . For a linear current we have

$$\mathbf{C} = \frac{J}{c\sqrt{|\mu|}} \oint \frac{d\mathbf{l}}{\sqrt{(\mu^{-1}_{ik} R_i R_k)}}.$$

### §31. Thermodynamic relations in a magnetic field

The thermodynamic relations for a magnetic substance in a magnetic field are, as we shall see, very similar to the corresponding relations for a dielectric in an electric field. Their

derivation, however, is quite different from that given in §10. This difference is ultimately due to the fact that a magnetic field, unlike an electric field, does no work on charges moving in it (since the force acting on a charge is perpendicular to its velocity). Hence, to calculate the change in the energy of the medium when a magnetic field is applied, we must examine the electric fields induced by the change in the magnetic field and determine the work done by these fields on the currents which produce the magnetic field.

Thus the equation which relates electric and variable magnetic fields must be used. This equation is

$$\text{curl } \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (31.1)$$

it follows immediately on averaging the microscopic equation (1.3).

During a time  $\delta t$ , the field  $\mathbf{E}$  does work  $\delta t \int \mathbf{j} \cdot \mathbf{E} dV$  on the currents  $\mathbf{j}$ . This quantity with the opposite sign is the work  $\delta R$  "done on the field" by the external e.m.f. which maintains the currents. Substituting

$$\mathbf{j} = c \text{curl } \mathbf{H}/4\pi,$$

we have

$$\begin{aligned} \delta R &= -\delta t \frac{c}{4\pi} \int \mathbf{E} \cdot \text{curl } \mathbf{H} dV \\ &= \delta t \frac{c}{4\pi} \int \text{div} (\mathbf{E} \times \mathbf{H}) dV - \delta t \frac{c}{4\pi} \int \mathbf{H} \cdot \text{curl } \mathbf{E} dV. \end{aligned}$$

The first integral, on being transformed to an integral over an infinitely distant surface, is seen to be zero. In the second integral we substitute  $\text{curl } \mathbf{E}$  from (31.1) and put  $\delta \mathbf{B} = \delta t \partial \mathbf{B}/\partial t$  for the change in the magnetic induction, obtaining finally

$$\delta R = \int \mathbf{H} \cdot \delta \mathbf{B} dV/4\pi. \quad (31.2)$$

This formula appears entirely analogous to the expression (10.2) for the work done in an infinitesimal change in the electric field. It must be pointed out, however, that the physical analogy between the two formulae is actually not complete, since  $\mathbf{H}$ , unlike  $\mathbf{E}$ , is not the mean value of the microscopic field.

Having derived formula (31.2), we can write down all the thermodynamic relations for a magnetic substance in a magnetic field by analogy with those given in §10 for a dielectric in an electric field, simply replacing  $\mathbf{E}$  and  $\mathbf{D}$  by  $\mathbf{H}$  and  $\mathbf{B}$  respectively. We shall give some of these formulae here for purposes of reference. The differentials of the total free energy and the total internal energy are

$$\left. \begin{aligned} \delta \mathcal{F} &= -\mathcal{S} \delta T + \int \mathbf{H} \cdot \delta \mathbf{B} dV/4\pi, \\ \delta \mathcal{U} &= T \delta \mathcal{S} + \int \mathbf{H} \cdot \delta \mathbf{B} dV/4\pi, \end{aligned} \right\} \quad (31.3)$$

and those of the corresponding quantities per unit volume are

$$\left. \begin{aligned} dF &= -S dT + \zeta d\rho + \mathbf{H} \cdot d\mathbf{B}/4\pi, \\ dU &= T dS + \zeta d\rho + \mathbf{H} \cdot d\mathbf{B}/4\pi. \end{aligned} \right\} \quad (31.4)$$

We need also the thermodynamic potentials

$$\tilde{U} = U - \mathbf{H} \cdot \mathbf{B}/4\pi, \quad \tilde{F} = F - \mathbf{H} \cdot \mathbf{B}/4\pi, \quad (31.5)$$

for which

$$\left. \begin{aligned} d\tilde{F} &= -SdT + \zeta d\rho - \mathbf{B} \cdot d\mathbf{H}/4\pi, \\ d\tilde{U} &= TdS + \zeta d\rho - \mathbf{B} \cdot d\mathbf{H}/4\pi. \end{aligned} \right\} \quad (31.6)$$

If the linear relation  $\mathbf{B} = \mu\mathbf{H}$  holds, we can write the expressions for all these quantities in the form

$$\left. \begin{aligned} U &= U_0(S, \rho) + B^2/8\pi\mu, & F &= F_0(T, \rho) + B^2/8\pi\mu, \\ \tilde{U} &= U_0(S, \rho) - \mu H^2/8\pi, & \tilde{F} &= F_0(T, \rho) - \mu H^2/8\pi. \end{aligned} \right\} \quad (31.7)$$

The work  $\delta R$  (or, what is the same thing, the change  $\delta\mathcal{F}$  at constant temperature) can be written in a different form, in terms of the current density and the vector potential of the magnetic field. For this purpose we put  $\delta\mathbf{B} = \text{curl } \delta\mathbf{A}$  and

$$\begin{aligned} (\delta\mathcal{F})_T &= \frac{1}{4\pi} \int \mathbf{H} \cdot \text{curl } \delta\mathbf{A} \, dV \\ &= -\frac{1}{4\pi} \int \text{div}(\mathbf{H} \times \delta\mathbf{A}) \, dV + \frac{1}{4\pi} \int \delta\mathbf{A} \cdot \text{curl } \mathbf{H} \, dV. \end{aligned}$$

The first integral is again zero, and the second gives

$$(\delta\mathcal{F})_T = \int \mathbf{j} \cdot \delta\mathbf{A} \, dV/c. \quad (31.8)$$

A similar transformation gives

$$(\delta\tilde{\mathcal{F}})_T = - \int \mathbf{A} \cdot \delta\mathbf{j} \, dV/c. \quad (31.9)$$

It is useful to note that in macroscopic electrodynamics the currents (sources of the magnetic field) are mathematically analogues of the potentials, not of the charges (the sources of the electric field). This is seen by comparing formulae (31.8) and (31.9) with the corresponding results for an electric field:

$$(\delta\mathcal{F})_T = \int \phi \delta\rho \, dV, \quad (\delta\tilde{\mathcal{F}})_T = - \int \rho \delta\phi \, dV \quad (31.10)$$

(see (10.13), (10.14)). We observe that the charges and potentials appear in these formulae in the opposite order to the currents and potentials in formulae (31.8), (31.9).†

On account of the complete formal correspondence between the thermodynamic relations (expressed in terms of field and induction) for electric and magnetic fields, the thermodynamic inequalities derived in §18 can also be applied to magnetic fields. In particular, we have seen that it follows from these inequalities that  $\epsilon > 0$ . In the electric case this result was of no interest, because it was weaker than the inequality  $\epsilon > 1$  which follows on other grounds. In the magnetic case, however, the corresponding inequality  $\mu > 0$  is very important, as it is the only restriction on the values which can be taken by the magnetic permeability.

† The significance of this difference is further discussed in the last footnote to §33.

### §32. The total free energy of a magnetic substance

In §11 expressions have been derived for the total free energy  $\mathcal{F}$  of a dielectric in an electric field. One of the thermodynamic properties of this quantity is that the change in it gives the work done by the electric field on the body when the charges producing the field remain constant. In a magnetic field a similar part is played by the free energy  $\mathcal{F}$ , since for given currents producing the field the change in  $\mathcal{F}$  is the work done on the body.

The following derivation is entirely analogous to that given in §11. The "total" quantity  $\mathcal{F}$  is defined as

$$\mathcal{F} = \int \left( \tilde{F} + \frac{\mathfrak{H}^2}{8\pi} \right) dV, \quad (32.1)$$

where  $\mathfrak{H}$  is the magnetic field which would be produced by the given currents in the absence of the magnetizable medium. The plus sign appears in the parenthesis (instead of the minus sign as in (11.1)) because the value of  $\mathcal{F}$  for a magnetic field in a vacuum is  $-\int (\mathfrak{H}^2/8\pi) dV$  (see (31.7)). The integration in (32.1) is taken over all space, including the volume occupied by the conductors in which flow the currents producing the field.†

Let us calculate the change in  $\mathcal{F}$  (for a given temperature and no departure from thermodynamic equilibrium in the medium) corresponding to an infinitesimal change in the field. Since  $\delta \tilde{F} = -\mathbf{B} \cdot \delta \mathbf{H}/4\pi$ , we have

$$\begin{aligned} \delta \mathcal{F} &= -\int (\mathbf{B} \cdot \delta \mathbf{H} - \mathfrak{H} \cdot \delta \mathfrak{H}) dV/4\pi \\ &= -\int (\mathbf{H} - \mathfrak{H}) \cdot \delta \mathfrak{H} dV/4\pi - \int \mathbf{B} \cdot (\delta \mathbf{H} - \delta \mathfrak{H}) dV/4\pi - \int (\mathbf{B} - \mathbf{H}) \cdot \delta \mathfrak{H} dV/4\pi. \end{aligned} \quad (32.2)$$

Introducing the vector potential  $\mathfrak{A}$  of the field  $\mathfrak{H}$ , we can write in the first term

$$\begin{aligned} (\mathbf{H} - \mathfrak{H}) \cdot \delta \mathfrak{H} &= (\mathbf{H} - \mathfrak{H}) \cdot \text{curl } \delta \mathfrak{A} \\ &= \text{div} [\delta \mathfrak{A} \times (\mathbf{H} - \mathfrak{H})] + \delta \mathfrak{A} \cdot \text{curl} (\mathbf{H} - \mathfrak{H}). \end{aligned}$$

By definition, the fields  $\mathbf{H}$  and  $\mathfrak{H}$  are produced by the same currents  $\mathbf{j}$ , the distribution of which over the volume of the conductors is (see §30) independent of the field which they produce, i.e. is independent of the presence or absence of magnetic substances in the surrounding medium. Hence  $\text{curl } \mathbf{H}$  and  $\text{curl } \mathfrak{H}$  are both equal to  $4\pi\mathbf{j}/c$ , and so  $\text{curl} (\mathbf{H} - \mathfrak{H}) = 0$ . The integral of  $\text{div} [\delta \mathfrak{A} \times (\mathbf{H} - \mathfrak{H})]$  is transformed into an integral over an infinitely distant surface, and so vanishes.

Similarly, we see that the second term on the right of (32.2) is zero; thus

$$\begin{aligned} \delta \mathcal{F} &= -\int (\mathbf{B} - \mathbf{H}) \cdot \delta \mathfrak{H} dV/4\pi \\ &= -\int \mathbf{M} \cdot \delta \mathfrak{H} dV. \end{aligned} \quad (32.3)$$

The expression which we have obtained for  $\delta \mathcal{F}$  is exactly similar to (11.3) for the electrostatic problem. In particular, in a uniform magnetic field  $\mathfrak{H}$  we have for  $d\mathcal{F}$  an

† In §11 we took the integration in (11.1) over all space except the volume occupied by the charged conductor producing the field. This was possible because there is no electric field in a conductor, charged or not. There is magnetic field, however, inside the conductors which carry the currents, and it cannot be excluded in calculating the total free energy.

expression analogous to (11.5):

$$d\tilde{\mathcal{F}} = -\mathcal{S} dT - \mathcal{M} \cdot d\mathfrak{H}, \quad (32.4)$$

where  $\mathcal{M}$  is the total magnetic moment of the body.

Without repeating the subsequent calculations, we shall write down the following formulae by analogy with those in §11. If the linear relation  $\mathbf{B} = \mu\mathbf{H}$  holds, we have

$$\tilde{\mathcal{F}} - \mathcal{F}_0(V, T) = -\int \frac{1}{2} \mathfrak{H} \cdot \mathbf{M} dV. \quad (32.5)$$

In particular, if the external field is homogeneous, then

$$\tilde{\mathcal{F}} - \mathcal{F}_0(V, T) = -\frac{1}{2} \mathfrak{H} \cdot \mathcal{M}. \quad (32.6)$$

In the general case of an arbitrary relation between  $\mathbf{B}$  and  $\mathbf{H}$ ,  $\tilde{\mathcal{F}}$  can be calculated from the formula

$$\begin{aligned} \tilde{\mathcal{F}} &= \int \left( \tilde{F} + \frac{\mathbf{H} \cdot \mathbf{B}}{8\pi} - \frac{1}{2} \mathbf{M} \cdot \mathfrak{H} \right) dV \\ &= \int \left( F - \frac{\mathbf{H} \cdot \mathbf{B}}{8\pi} - \frac{1}{2} \mathbf{M} \cdot \mathfrak{H} \right) dV, \end{aligned} \quad (32.7)$$

which is analogous to (11.12) for dielectrics.

In §11 we gave also the simpler formulae obtained when the dielectric susceptibility is small. The analogous case for the magnetic problem is especially important because, as mentioned above, the magnetic susceptibility of the majority of bodies is indeed small. In this case

$$\tilde{\mathcal{F}} - \mathcal{F}_0 = -\frac{1}{2} \chi \int \mathfrak{H}^2 dV. \quad (32.8)$$

We can also derive results for the magnetic field analogous to those obtained in §14. These concern the change in the thermodynamic quantities resulting from an infinitesimal change in the magnetic permeability  $\mu$ , the field sources being assumed unchanged. It is clear from the foregoing that we must consider the change in  $\tilde{\mathcal{F}}$ , and not that in  $\mathcal{F}$  as in §14. We shall not repeat the derivation, which is similar to that of (14.1), but merely give the result:

$$\delta\tilde{\mathcal{F}} = -\int \delta\mu H^2 dV/8\pi. \quad (32.9)$$

In §14 we used the formula (11.7), an analogue of (32.5), to deduce that the dielectric susceptibility of any substance is positive. In the magnetic case we cannot draw this conclusion, and the magnetic susceptibility may be of either sign. The reason for this marked difference is that the Hamiltonian of a system of charges moving in a magnetic field contains not only terms linear in the field (as in the electric case) but also quadratic terms. Hence, in determining the change in the free energy of the body in the magnetic field by means of perturbation theory as in (14.2), we have a contribution in the first approximation as well as the second. In such a case no general conclusion can be drawn concerning the sign of the variation. It is positive for paramagnetic bodies and negative for diamagnetic ones.

In §14 we also drew conclusions concerning the direction of motion of bodies in an electric field. Similar conclusions follow from (32.9), but, since  $\mu$  may be either greater or less than 1, there is no universal result. For example, in an almost uniform field paramagnetic bodies ( $\mu > 1$ ) move in the direction of  $H$  increasing, and diamagnetic bodies ( $\mu < 1$ ) in the opposite direction.

### §33. The energy of a system of currents

Let us consider a system of conductors with currents flowing in them and assume that neither the conductors nor the medium surrounding them are ferromagnetic, so that  $\mathbf{B} = \mu\mathbf{H}$  everywhere. According to §31, the total free energy of the system is given in terms of the magnetic field of the currents by

$$\mathcal{F} = \int \mathbf{H} \cdot \mathbf{B} dV / 8\pi. \quad (33.1)$$

Here we omit the quantity  $\mathcal{F}_0$ , which is a constant (at a given temperature) and is not related to the currents. The integration in (33.1) is taken over all space, both inside and outside the conductors.

The same energy can also be expressed in terms of the currents by means of the integral

$$\mathcal{F} = \int \mathbf{A} \cdot \mathbf{j} dV / 2c; \quad (33.2)$$

cf. the derivation of (31.8) from (31.2). Here the integration extends only over the conductors, because  $\mathbf{j} = 0$  outside them.

Since the field equations are linear, the magnetic field can be written as the sum of the fields resulting from each current alone with no current in the other conductors:  $\mathbf{H} = \sum \mathbf{H}_a$ . Then the total free energy (33.1) is

$$\mathcal{F} = \sum_a \mathcal{F}_{aa} + \sum_{a>b} \mathcal{F}_{ab}, \quad (33.3)$$

where

$$\mathcal{F}_{aa} = \int \mathbf{H}_a \cdot \mathbf{B}_a dV / 8\pi, \quad \mathcal{F}_{ab} = \int \mathbf{H}_a \cdot \mathbf{B}_b dV / 4\pi. \quad (33.4)$$

We have put  $\mathcal{F}_{ab} = \mathcal{F}_{ba}$ , since  $\mathbf{H}_a \cdot \mathbf{B}_b = \mu \mathbf{H}_a \cdot \mathbf{H}_b = \mathbf{H}_b \cdot \mathbf{B}_a$ , where  $\mu$  is the magnetic permeability at any point. The quantity  $\mathcal{F}_{aa}$  may be called the *free self-energy* of the current in the  $a$ th conductor, and  $\mathcal{F}_{ab}$  the *interaction energy* of the  $a$ th and  $b$ th conductors. It should be borne in mind, however, that these names are strictly correct only if the magnetic properties of both the conductors and the medium are neglected. Otherwise the field, and therefore the energy, of each current depend on the position and magnetic permeability of the other conductors.

The quantities (33.4) can also be expressed in terms of the currents  $\mathbf{j}_a$  in each conductor, in accordance with formula (33.2):

$$\mathcal{F}_{aa} = \int \mathbf{j}_a \cdot \mathbf{A}_a dV_a / 2c, \quad \mathcal{F}_{ab} = \int \mathbf{j}_a \cdot \mathbf{A}_b dV_a / c = \int \mathbf{j}_b \cdot \mathbf{A}_a dV_b / c. \quad (33.5)$$

The integral in  $\mathcal{F}_{aa}$  is here taken only over the volume of the  $a$ th conductor;  $\mathcal{F}_{ab}$  can be written as either of the two expressions, in which the integration is over the volume of the  $a$ th and  $b$ th conductor respectively.

When the distribution of the current density over the volume of the conductor is given,  $\mathcal{F}_{aa}$  depends only on the total current  $J_a$  passing through a cross-section. Both the current density  $\mathbf{j}$  and the field which it produces will be proportional to  $J_a$ . Hence the integral  $\mathcal{F}_{aa}$  is proportional to  $J_a^2$ , and we write it

$$\mathcal{F}_{aa} = L_{aa} J_a^2 / 2c^2, \quad (33.6)$$

where  $L_{aa}$  is called the *self-inductance* of the conductor. Similarly, the interaction energy of two currents is proportional to the product  $J_a J_b$ :

$$\mathcal{F}_{ab} = L_{ab} J_a J_b / c^2. \quad (33.7)$$



The quantity  $L_{ab}$  is called the *mutual inductance* of the conductors. Thus the total free energy of a system of currents is

$$\mathcal{F} = \frac{1}{2c^2} \sum_a L_{aa} J_a^2 + \frac{1}{c^2} \sum_{a>b} L_{ab} J_a J_b = \frac{1}{2c^2} \sum_a \sum_b L_{ab} J_a J_b. \quad (33.8)$$

The condition that this quadratic form should be positive definite places certain restrictions on the values of the coefficients. In particular  $L_{aa} > 0$  for all  $a$ , and  $L_{aa}L_{bb} > L_{ab}^2$ .

The calculation of the energy of currents in the general case of arbitrary three-dimensional conductors requires a complete solution of the field equations, and is a difficult problem. It becomes simpler if the magnetic permeability of both the conductors and the surrounding medium can be taken as unity. It should be noted that the energy of the currents is then no longer dependent on the thermodynamic state (in particular, on the temperature) of the bodies, and hence the free energy in the above formulae may be referred to simply as the energy.

For  $\mu = 1$  the vector field potential due to the currents  $\mathbf{j}$  is given by formula (30.12). Hence the self-energy of the  $a$ th conductor is

$$\mathcal{F}_{aa} = \frac{1}{2c^2} \int \int \frac{\mathbf{j} \cdot \mathbf{j}'}{R} dV dV', \quad (33.9)$$

where both integrations are taken over the volume of the conductor considered, and  $R$  is the distance between  $dV$  and  $dV'$ . Similarly, the mutual energy of two conductors is

$$\mathcal{F}_{ab} = \frac{1}{c^2} \int \int \frac{\mathbf{j}_a \cdot \mathbf{j}_b}{R} dV_a dV_b, \quad (33.10)$$

where  $dV_a$  and  $dV_b$  are volume elements in the two conductors.

The mutual energy of two linear currents is particularly easy to calculate. In formula (33.10) we change from volume currents to linear ones by replacing  $\mathbf{j}_a dV_a$  and  $\mathbf{j}_b dV_b$  by  $J_a d\mathbf{l}_a$  and  $J_b d\mathbf{l}_b$  respectively, and we find that the mutual inductance is  $L_{ab} = \oint \oint d\mathbf{l}_a \cdot d\mathbf{l}_b / R$ . In this approximation, therefore,  $L_{ab}$  depends only on the shape, size and relative position of the two currents, and not on the distribution of current over the cross-section of each wire. It must be emphasized that this simple formula can be obtained for linear currents without imposing the condition that  $\mu = 1$ . In the approximation where the thickness of the wires is neglected, their magnetic properties have no effect on the field which they produce, and therefore no effect on their mutual energy. If the magnetic permeability  $\mu$  of the medium surrounding the wires is different from unity, the vector potential is, by (30.15), simply multiplied by  $\mu$ , and therefore so is the magnetic induction. The mutual inductance is therefore multiplied by the same factor, so that

$$L_{ab} = \mu \oint \oint d\mathbf{l}_a \cdot d\mathbf{l}_b / R. \quad (33.11)$$

The self-inductance of linear conductors is much more difficult to calculate; we shall discuss it in §34.

The total energy of a system of linear currents can be written in still another form. To do this, we return to the integral (33.2), which for linear currents becomes

$$\mathcal{F} = \frac{1}{2c} \sum_a J_a \oint \mathbf{A} \cdot d\mathbf{l}_a, \quad (33.12)$$

where  $\mathbf{A}$  is the vector potential of the total field at the element  $d\mathbf{l}_a$  of the  $a$ th conductor. The main error in going from (33.2) to (33.12) arises from neglecting the change in the field (including the field of the current considered) over the cross-section of the wire. Each of the contour integrals in (33.12) can be transformed into a surface integral:

$$\oint \mathbf{A} \cdot d\mathbf{l}_a = \int \text{curl } \mathbf{A} \cdot d\mathbf{f}_a = \int \mathbf{B} \cdot d\mathbf{f}_a,$$

i.e. it is the flux of the magnetic induction or *magnetic flux* through the circuit of the  $a$ th current. We denote this flux by  $\Phi_a$ . Then

$$\mathcal{F} = \frac{1}{2c} \sum_a J_a \Phi_a. \quad (33.13)$$

Similarly, the free energy  $\mathcal{F}$  of a linear current  $J$  in an external magnetic field, i.e. the energy without the self-energy of the field sources, can be expressed in terms of the magnetic flux. Evidently

$$\mathcal{F} = J\Phi/c, \quad (33.14)$$

where  $\Phi$  is the flux of the external field through the circuit of the current  $J$ . If the external field is uniform, and  $\mu = 1$  in the external medium, then  $\Phi = \mathfrak{H} \cdot \int d\mathbf{f}$ . Introducing the magnetic moment of the current in accordance with (30.18), we have  $\mathcal{F} = \mathcal{M} \cdot \mathfrak{H}$ .

Knowing the energy of a system of currents as a function of their shape, size and relative position, we can determine the forces on the conductors by simply differentiating with respect to the appropriate coordinates. Here, however, the question arises which characteristics of the currents should be kept constant in the differentiation. It is most convenient to differentiate at constant current. In this case the free energy is represented by  $\mathcal{F}$ , and so the generalized force  $F_q$  in the direction of a generalized coordinate  $q$  is  $F_q = -(\partial \mathcal{F} / \partial q)_{J,T}$ . The suffixes show that the differentiation is effected at constant current and constant temperature. Since we omit the term independent of the currents in the free energy,  $\mathcal{F}$  and  $\tilde{\mathcal{F}}$  differ only in sign, and so

$$F_q = -\left(\frac{\partial \tilde{\mathcal{F}}}{\partial q}\right)_J = \left(\frac{\partial \mathcal{F}}{\partial q}\right)_J = \frac{1}{2c^2} \sum_{a,b} J_a J_b \frac{\partial L_{ab}}{\partial q}; \quad (33.15)$$

here and henceforward the suffix  $T$  to the derivatives is omitted, for brevity.

In particular, the forces exerted on a conductor by its own magnetic field are given by the formula

$$F_q = \frac{1}{2c^2} J^2 \frac{\partial L}{\partial q}, \quad (33.16)$$

where  $L$  is the self-inductance of the conductor. The nature of these forces can be seen as follows. For given current (and temperature),  $\tilde{\mathcal{F}}$  tends to a minimum. Since  $\tilde{\mathcal{F}} = -LJ^2/2c^2$ , this means that the forces on the conductor will tend to increase its self-inductance. The latter, having the dimensions of length, must be proportional to the dimension of the conductor. Thus the effect of the magnetic field is to increase the size of the conductor.

For a current in an external magnetic field we have†

$$\tilde{\mathcal{F}} = -\mathcal{F} = -\mathcal{M} \cdot \mathfrak{H}. \quad (33.17)$$

† The factor  $\frac{1}{2}$  which appears in (32.6) is absent in (33.17) because the magnetic moment of the current in the latter equation is independent of the field, whereas the magnetic moment in (32.6) is itself due to the field.

In all the above formulae for the energy it is assumed that there is a linear relation between the magnetic field and induction. In the general case where this relation is arbitrary, analogous differential relations can be set up. The change in the free energy resulting from an infinitesimal change in the field (at constant temperature) is, by (31.8),  $\delta\mathcal{F} = \int \mathbf{j} \cdot \delta\mathbf{A} \, dV/c$  or, for a system of linear currents,

$$\delta\mathcal{F} = \frac{1}{c} \sum_a J_a \oint \delta\mathbf{A} \cdot d\mathbf{l}_a.$$

Proceeding as in the derivation of (33.13) from (33.12), we have†

$$\delta\mathcal{F} = \frac{1}{c} \sum_a J_a \delta\Phi_a. \quad (33.18)$$

Similarly, we find from (31.9)

$$\delta\tilde{\mathcal{F}} = -\frac{1}{c} \sum_a \Phi_a \delta J_a. \quad (33.19)$$

Thus we can say that, for a system of linear currents,  $\mathcal{F}$  is the thermodynamic potential with respect to the magnetic fluxes, and  $\tilde{\mathcal{F}}$  with respect to the currents, the two potentials being related by

$$\tilde{\mathcal{F}} = \mathcal{F} - \frac{1}{c} \sum_a J_a \Phi_a. \quad (33.20)$$

Whatever the magnetic properties of the substance, therefore, the thermodynamic relations

$$J_a/c = \partial\mathcal{F}/\partial\Phi_a, \quad \Phi_a/c = -\partial\tilde{\mathcal{F}}/\partial J_a \quad (33.21)$$

hold. If these formulae are applied to the case where the field and induction are linearly related, so that  $\mathcal{F}$  is given by (33.8), we obtain

$$\Phi_a = \frac{1}{c} \sum_b L_{ab} J_b. \quad (33.22)$$

Thus the inductances are the coefficients of proportionality between the magnetic fluxes and the currents which produce the magnetic field. The product  $L_{ab} J_b/c$  is the magnetic flux through the circuit of the current  $J_a$  due to the current  $J_b$  ( $b \neq a$ ), and  $L_{aa} J_a/c$  is that due to the current  $J_a$  itself.

### §34. The self-inductance of linear conductors

In calculating the self-inductance of a linear conductor its thickness cannot be entirely neglected as it was in calculating the mutual inductance of two conductors. If it were, we

† There is an obvious analogy between formulae (33.18) and (10.13) in the magnetic and electric cases respectively. In the magnetic case, the induction fluxes play the role of charges. The analogy has a clear physical interpretation. Just as the electric field can be maintained without any external energy supply, by charges on insulated conductors, the magnetic field can be maintained, without any external energy supply, by superconducting solenoids across which the magnetic fluxes remain constant. It is therefore not surprising that the change in the free energy  $\mathcal{F}$  in the electric and magnetic cases is governed by the changes in the charges and in the induction fluxes respectively.

should obtain from (33.9) the self-inductance  $L = \oint \oint d\mathbf{l} \cdot d\mathbf{l}'/R$ , where both integrals are taken along the same circuit, and this integral is logarithmically divergent as  $R \rightarrow 0$ .

The exact value of the self-inductance of a conductor depends on the distribution of current in it, which may vary with the manner of excitation of the current, i.e. with the manner of application of the electromotive force. For a linear conductor, however, the self-inductance does not, to a fairly high accuracy, depend on the distribution of current over the cross-section.†

Let us write the self-inductance as  $L = L_e + L_i$ , where  $L_e$  and  $L_i$  result from the magnetic field energy outside and inside the conductor respectively. For a linear conductor, the "external" part  $L_e$  makes the main contribution to the self-inductance. This is because most of the magnetic energy of a closed linear circuit resides in the field at distances from the wire large compared with its thickness. For the energy per unit length of an infinite straight wire is

$$(\mu_e/8\pi) \int H^2 \cdot 2\pi r dr = (\mu_e/8\pi) \int (2J/cr)^2 \cdot 2\pi r dr = (\mu_e J^2/c^2) \int dr/r,$$

where  $r$  is the distance from the axis of the wire and  $\mu_e$  the magnetic permeability of the external medium. This integral diverges logarithmically for large  $r$ . For a closed linear circuit, of course, this divergence disappears, because the integral is "cut off" at distances of the order of the dimension of the circuit. We obtain an approximate value for the energy on multiplying this integral by the total length  $l$  of the wire, and taking  $l$  as the upper limit and the radius  $a$  of the wire as the lower limit. The result is  $(\mu_e J^2 l/c^2) \log(l/a)$ , and hence the self-inductance is

$$L = 2\mu_e l \log(l/a). \quad (34.1)$$

This expression is said to be of *logarithmic accuracy*: its relative error is of the order of  $1/\log(l/a)$ , and the ratio  $l/a$  is assumed to be so large that its logarithm is large.‡

A particular case of a linear conductor is a *solenoid*, which consists of a wire wound in a helix, with the turns very close together. Neglecting the thickness of the wire and the distance between the turns, we have simply a conducting cylindrical surface with a "surface" conduction current on it. The equation  $\text{curl } \mathbf{H} = 4\pi \mathbf{j}/c$  within the conductor is here replaced by the boundary condition.

$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = 4\pi \mathbf{g}/c, \quad (34.2)$$

where  $\mathbf{g}$  is the surface current density,  $\mathbf{H}_1$  and  $\mathbf{H}_2$  the fields on each side of the surface, and  $\mathbf{n}$  the unit normal vector into medium 2; cf. the derivation of (29.16).

If the solenoid is an infinitely long cylinder, the magnetic field which it produces can be found very simply. The surface currents flow in circles, and their density  $g = nJ$ , where  $J$  is the current in the wire and  $n$  the number of turns per unit length of the solenoid. The field outside the cylinder is zero; the field inside is uniform and along the axis of the cylinder, and

† More precisely, it is independent of the distribution of current provided that the current density varies appreciably only over distances comparable with the thickness  $a$  of the wire. If, however, the distribution is such that the current density varies appreciably over distances small compared with  $a$  (as happens, for particular reasons, in the *skin effect* and in superconductors), then the self-inductance does depend on the distribution.

‡ The assertion made above that the self-inductance is independent of the current distribution actually applies not only to the approximation (34.1) but also to the next approximation, in which terms not containing the large logarithm are included (or, what is the same thing, the coefficient of  $l/a$  is included in the argument of the logarithm); see the Problems at the end of this section.

is  $H = 4\pi nJ/c$ . For this field evidently satisfies the equations  $\text{div } \mathbf{H} = 0$ ,  $\text{curl } \mathbf{H} = 0$  in all space outside the conducting surface, and also the boundary condition (34.2) at that surface.

Accordingly, the field energy per unit length of the cylinder is

$$\mu_e H^2 \pi b^2 / 8\pi = 2\pi^2 n^2 b^2 \mu_e J^2 / c^2,$$

where  $b$  is the radius of the cylinder and  $\mu_e$  pertains to the material within the solenoid. Neglecting the end effects, we can apply this formula also to a solenoid whose length  $h$  is finite, but large compared with  $b$ . Then the self-inductance is

$$L = 4\pi^2 n^2 b^2 h \mu_e = 2\pi \mu_e n b l, \quad (34.3)$$

where  $l = 2\pi b n h$  is the total length of the wire. The greater self-inductance of a solenoid as compared with that of a straight wire of equal length (cf. (34.1)) is, of course, due to the mutual induction between adjoining turns.

### PROBLEMS†

PROBLEM 1. Determine the self-inductance of a closed circuit of thin wire of circular cross-section.

SOLUTION. The magnetic field in the wire can be taken to be the same as that inside an infinite straight cylinder:  $H = 2Jr/ca^2$ , where  $r$  is the distance from the axis of the wire and  $a$  its radius. Hence we find the internal part of the self-inductance:

$$L_i = \frac{2c^2 \mu_i}{J^2 8\pi} \int H^2 dV = \frac{1}{2} l \mu_i, \quad (1)$$

where  $l$  is the length of the wire.

To calculate  $L_e$ , we notice that the field outside a thin wire is independent of the distribution of current over its cross-section. In particular, the energy  $\mathcal{F}_e$  of the external magnetic field is unchanged if we assume that the current flows only on the surface of the wire. The field inside the wire is then zero, and  $\mathcal{F}_e$  may be calculated as the total energy from formula (33.2). On account of the assumed surface distribution of the current, the integral in this formula becomes a line integral along the axis of the wire, and so the external part of the self-inductance is

$$L_e = \frac{2c^2 J}{J^2 2c} \oint [\mathbf{A}]_{r=a} \cdot d\mathbf{l},$$

where the value of  $\mathbf{A}$  in the integrand is taken at the surface of the wire. In obtaining this formula we also use the fact that, in the approximation used here, the field is constant along the perimeter of a cross-section.

Having reduced the problem to that of calculating  $\mathbf{A}$  for  $r = a$ , we now make a different assumption concerning the current distribution, namely that the whole current  $J$  flows along the axis of the wire. The field on the surface of the wire is, in the approximation considered, unchanged by this assumption (nor would it be changed for a straight wire of circular cross-section). Then, by formula (30.14), we have

$$[\mathbf{A}]_{r=a} = \frac{J}{c} \left[ \oint \frac{d\mathbf{l}}{R} \right]_{r=a},$$

where  $R$  is the distance from the element  $d\mathbf{l}$  of the axis to a given point on the surface of the wire. We divide the integral into two parts, one for which  $R > \Delta$  and the other for which  $R < \Delta$ , where  $\Delta$  is a distance small compared with the dimension of the circuit but large compared with the radius  $a$  of the wire.‡ In the integral where  $R > \Delta$ ,  $a$  may be neglected and  $R$  taken simply as the distance between two points on the circuit. The vector integral where  $R < \Delta$  may be assumed to be along the tangent at the point considered. Denoting by  $\mathbf{t}$  the unit vector in that direction, we have

$$\begin{aligned} \left[ \int_{R < \Delta} \frac{d\mathbf{l}}{R} \right]_{r=a} &\cong \mathbf{t} \int_{-\Delta}^{\Delta} \frac{dl}{\sqrt{(a^2 + l^2)}} = 2\mathbf{t} \sinh^{-1}(\Delta/a) \\ &\cong 2\mathbf{t} \log(2\Delta/a). \end{aligned}$$

† In Problems 1–6 we assume  $\mu_e = 1$ .

‡ A similar procedure was used to calculate the capacitance of a thin ring in §2, Problem 4.

This expression can be written as the integral

$$\int_{\Delta > R > \frac{1}{2}a} d\mathbf{l}/R,$$

where  $R$  is again the distance between points on the circuit. Adding the two integrals for  $R > \Delta$  and  $R < \Delta$ , we obtain

$$[\mathbf{A}]_{r=a} = \frac{J}{c} \int_{R > \frac{1}{2}a} \frac{d\mathbf{l}}{R},$$

from which the arbitrary parameter  $\Delta$  has disappeared, as it should.

The final result is therefore

$$L_e = \iint_{R > \frac{1}{2}a} \frac{d\mathbf{l} \cdot d\mathbf{l}'}{R}. \quad (2)$$

The integration here extends over all pairs of points on the circuit whose distance apart exceeds  $\frac{1}{2}a$ .

**PROBLEM 2.** Determine the self-inductance of a thin wire ring (with radius  $b$ ) of circular cross-section (with radius  $a$ ).

**SOLUTION.** The integrand in (2), Problem 1, depends only on the central angle  $\phi$  subtended by the chord  $R$ , and  $R = 2b \sin \frac{1}{2}\phi$ , while  $d\mathbf{l} \cdot d\mathbf{l}' = dl dl' \cos \phi$ . Hence

$$L_e = 2 \int_{\phi_0}^{\pi} \frac{\cos \phi \cdot 2\pi b \cdot b d\phi}{2b \sin \frac{1}{2}\phi} = 4\pi b \left[ -\log \tan \frac{1}{4}\phi_0 - 2 \cos \frac{1}{2}\phi_0 \right].$$

The lower limit of integration is determined from  $2b \sin \frac{1}{2}\phi_0 = \frac{1}{2}a$ , whence  $\phi_0 \cong a/2b$ . Substituting this value and adding  $L_i = \pi b \mu_i$ , we have to the required accuracy

$$L = 4\pi b \left[ \log (8b/a) - 2 + \frac{1}{4}\mu_i \right].$$

In particular, for  $\mu_i = 1$  we obtain

$$L = 4\pi b \left[ \log (8b/a) - (7/4) \right].$$

**PROBLEM 3.** Determine the extension of a ring of wire (with  $\mu_i = 1$ ) under the action of the magnetic field of a current flowing in it.

**SOLUTION.** The internal stresses parallel and perpendicular to the axis are, by (33.16), given by

$$\pi a^2 \sigma_{\parallel} = \frac{J^2}{2c^2} \frac{\partial L}{\partial (2\pi b)}, \quad 2\pi a b \sigma_{\perp} = \frac{J^2}{2c^2} \frac{\partial L}{\partial a}.$$

Substituting  $L$  from Problem 2, we have

$$\sigma_{\parallel} = \frac{J^2}{\pi a^2 c^2} \left[ \log \frac{8b}{a} - \frac{3}{4} \right], \quad \sigma_{\perp} = -\frac{J^2}{a^2 c^2}.$$

Hence the required relative extension of the ring is

$$\frac{\Delta b}{b} = \frac{1}{E} (\sigma_{\parallel} - 2\sigma_{\perp}) = \frac{J^2}{\pi a^2 c^2 E} \left( \log \frac{8b}{a} - \frac{3}{4} + 2\pi \right).$$

where  $E$  is Young's modulus and  $\sigma$  Poisson's ratio for the wire; see *TE*, §5.

**PROBLEM 4.** Determine the self-inductance per unit length of a system of two parallel straight wires (with  $\mu_i = 1$ ) having circular cross-sections of radii  $a$  and  $b$ , with their axes a distance  $h$  apart, and carrying equal currents  $J$  in opposite directions (Fig. 19).

**SOLUTION.** The vector potential of the magnetic field of each current is parallel to the axes of the wires, and so the two vector potentials can be added algebraically. For the magnetic field of wire 1, with a uniformly distributed

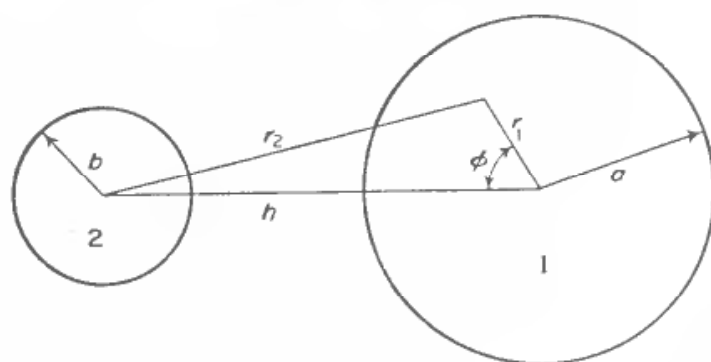


FIG. 19

current  $+J$ , we have in cylindrical polar coordinates

$$A_z = \frac{J}{c} \left( C - \frac{r^2}{a^2} \right) \text{ for } r < a,$$

$$A_z = \frac{J}{c} \left( C - 1 - 2 \log \frac{r}{a} \right) \text{ for } r > a,$$

where  $C$  is an arbitrary constant;  $A_z$  is continuous at the surface of the wire. The formulae for wire 2 are obtained by substituting  $b$  for  $a$  and changing the sign of  $J$ . Integration over the cross-section of wire 1 in formula (33.2) gives

$$\begin{aligned} & \frac{J^2}{2c^2\pi a^2} \int \left\{ \left( C - \frac{r_1^2}{a^2} \right) - \left( C - 1 - 2 \log \frac{r_2}{b} \right) \right\} df_1 \\ &= \frac{J^2}{2c^2\pi a^2} \int_0^a \int_0^{2\pi} \left\{ 1 - \frac{r_1^2}{a^2} + \log \frac{h^2 + r_1^2 - 2hr_1 \cos \phi}{b^2} \right\} r_1 d\phi dr_1 = \frac{J^2}{2c^2} \left( \frac{1}{2} + 2 \log \frac{h}{b} \right). \end{aligned}$$

The integration over the cross-section of wire 2 gives the same thing with  $a$  in place of  $b$ . The required self-inductance per unit length of the double wire is therefore

$$L = 1 + 2 \log(h^2/ab).$$

**PROBLEM 5.** Determine the self-inductance of a toroidal solenoid.

**SOLUTION.** We regard the solenoid as a toroidal conducting surface carrying surface currents with density  $g = NJ/2\pi r$ , where  $N$  is the total number of turns and  $J$  the current; the coordinates and dimensions are as shown in Fig. 20 (p. 126). The magnetic field outside the solenoid is zero, and inside the solenoid  $H_{ir} = H_{iz} = 0$ ,  $H_{i\phi} = 2NJ/cr$ , where  $r, z, \phi$  are cylindrical polar coordinates; for this solution satisfies the equations  $\text{div } \mathbf{H} = 0$ ,  $\text{curl } \mathbf{H} = 0$  and the boundary condition (34.2).† The energy of the magnetic field in the solenoid is

$$\int (H_i^2/8\pi) dV = (N^2 J^2/c^2) \oint z dr/r,$$

where the integration is taken along the perimeter of the cross-section, and is easily effected by putting  $z = a \sin \theta$ ,  $r = b + a \cos \theta$ . The self-inductance is found to be

$$L = 4\pi N^2 [b - \sqrt{(b^2 - a^2)}].$$

**PROBLEM 6.** Determine the end-effect correction of order  $l/h$  to the expression (34.3) (with  $\mu_e = 1$ ) for the self-inductance of a cylindrical solenoid.

**SOLUTION.** The self-inductance is calculated as a double integral over the surface of the solenoid:

$$L = \frac{1}{J^2} \iint \frac{\mathbf{g}_1 \cdot \mathbf{g}_2}{R} df_1 df_2,$$

† It is valid also for an annular solenoid with any cross-section.

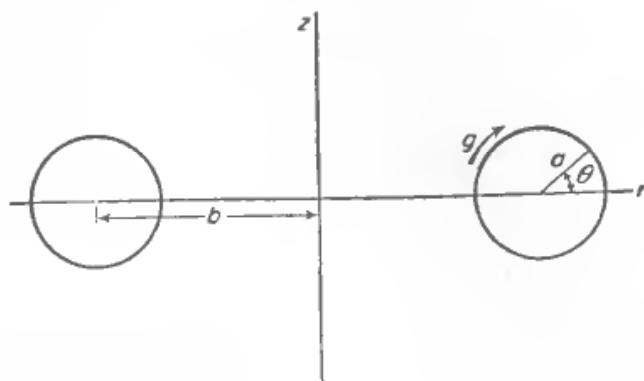


FIG. 20

where  $g$  is the surface current density ( $g = nJ$ ). In cylindrical polar coordinates

$$L = 2\pi b^2 n^2 \int_0^h \int_0^h \int_0^{2\pi} \frac{\cos \phi \, d\phi \, dz_1 \, dz_2}{\sqrt{[(z_2 - z_1)^2 + 4b^2 \sin^2 \frac{1}{2}\phi]}}$$

$$= 8\pi b^2 n^2 \int_0^h \int_0^\pi \frac{(h - \zeta) \cos \phi \, d\phi \, d\zeta}{\sqrt{(\zeta^2 + 4b^2 \sin^2 \frac{1}{2}\phi)}}$$

where  $\phi$  is the angle between the diametral planes through  $df_1$  and  $df_2$ , and  $\zeta = z_2 - z_1$ .

Effecting the integration with respect to  $\zeta$ , we have for  $h \gg b$

$$L \cong 8\pi b^2 n^2 \int_0^\pi [h \log \frac{h}{b \sin \frac{1}{2}\phi} - h + 2b \sin \frac{1}{2}\phi] \cos \phi \, d\phi,$$

and finally

$$L = 4\pi^2 b^2 n^2 [h - 8b/3\pi].$$

**PROBLEM 7.** Determine the factor by which the self-inductance of a plane circuit changes when it is placed on the surface of a half-space of magnetic permeability  $\mu_e$ . The internal part of the self-inductance is neglected.

**SOLUTION.** It is evident from symmetry that, in the absence of the half-space, the magnetic field of the current is symmetrical about the plane of the circuit, and the lines of magnetic force cross that plane normally. Let this field be  $H_0$ . We can satisfy the field equations and the boundary conditions on the surface of the half-space by putting  $H = 2\mu_e H_0/(\mu_e + 1)$  in the vacuum and  $B = \mu_e H = 2\mu_e H_0/(\mu_e + 1)$  in the medium:  $B_n$  and  $H_t$  are then continuous at the boundary, and the circulation of  $H$  along any line of force is equal to that of  $H_0$ . Hence we easily see that, when the medium is present, the total energy of the field, and therefore the self-inductance of the circuit, are multiplied by  $2\mu_e/(\mu_e + 1)$ .

### §35. Forces in a magnetic field

To determine the forces on matter in a magnetic field hardly any further calculations are necessary, on account of the complete analogy with electrostatics. The analogy is due mainly to the fact that the expressions for the thermodynamic quantities in a magnetic field differ from those for an electric field only in that  $E$  and  $D$  are replaced by  $H$  and  $B$  respectively. In calculating the stress tensor in §15 we used the fact that the electric field satisfies the equation  $\text{curl } E = 0$ , and is therefore a potential field. The magnetic field satisfies the equation

$$\text{curl } H = 4\pi j/c, \quad (35.1)$$



which reduces to  $\text{curl } \mathbf{H} = 0$  only in the absence of conduction currents. In calculating the stress tensor, however, we must always put  $\mathbf{j} = 0$ . Since  $\mathbf{j}$  involves the derivatives of the magnetic field, an allowance for the currents in calculating the stresses would amount to adding to the stress tensor  $\sigma_{ik}$  the very small corrections due to the non-uniformity of the field; cf. the second footnote to §15.

Thus all the formulae obtained in §§15 and 16 for the stress tensor can be applied immediately for a magnetic field. For example, in a fluid medium with  $\mathbf{B} = \mu\mathbf{H}$  we have

$$\sigma_{ik} = -P_0(\rho, T)\delta_{ik} - \frac{H^2}{8\pi} \left[ \mu - \rho \left( \frac{\partial \mu}{\partial \rho} \right)_T \right] \delta_{ik} + \frac{\mu H_i H_k}{4\pi}. \quad (35.2)$$

From this the body forces are calculated by the formula  $f_i = \delta \sigma_{ik} / \partial x_k$ . If the medium is a conductor carrying a current, the calculation differs from that in §15 in that the equation  $\text{curl } \mathbf{H} = 0$  is replaced by (35.1).

Differentiating (35.2) and using also the equation  $\text{div } \mathbf{B} = \text{div } (\mu\mathbf{H}) = 0$ , we find

$$\mathbf{f} = -\text{grad } P_0 + \frac{1}{8\pi} \text{grad} \left[ H^2 \rho \left( \frac{\partial \mu}{\partial \rho} \right)_T \right] - \frac{H^2}{8\pi} \text{grad } \mu - \frac{\mu}{8\pi} \text{grad } H^2 + \frac{\mu}{4\pi} (\mathbf{H} \cdot \text{grad}) \mathbf{H}.$$

By a well-known formula of vector analysis,

$$\begin{aligned} (\mathbf{H} \cdot \text{grad}) \mathbf{H} &= \frac{1}{2} \text{grad } H^2 - \mathbf{H} \times \text{curl } \mathbf{H} \\ &= \frac{1}{2} \text{grad } H^2 + 4\pi \mathbf{j} \times \mathbf{H} / c. \end{aligned}$$

Thus

$$\mathbf{f} = -\text{grad } P_0 + \frac{1}{8\pi} \text{grad} \left[ H^2 \rho \left( \frac{\partial \mu}{\partial \rho} \right)_T \right] - \frac{H^2}{8\pi} \text{grad } \mu + \frac{\mu}{c} \mathbf{j} \times \mathbf{H}. \quad (35.3)$$

The last term does not appear in the corresponding formula (15.12). It would, however, be incorrect to suppose that the presence of this term means that a force can be isolated in  $\mathbf{f}$  which is due to the conduction current. The reason is that, by (35.1), the current  $\mathbf{j}$  is inseparable from non-uniformity of the field, and another term in (35.3) also involves the space derivatives of the field. When the magnetic permeability of the medium is appreciably different from unity, all the terms in (35.3) are in general of the same order of magnitude.

If, however, as usually happens,  $\mu$  is close to 1, the last term in (35.3) gives the main contribution to the force when a conduction current is present, and the remaining terms form only a small correction. In calculating the forces we can then put  $\mu = 1$ , obtaining simply

$$\mathbf{f} = \mathbf{j} \times \mathbf{H} / c. \quad (35.4)$$

The term  $-\text{grad } P_0$  is of no interest henceforward, and we omit it. For  $\mu = 1$  the properties of the substance have no effect on the magnetic phenomena, and the expression (35.4) for the force is equally valid for fluid and for solid conductors. The total force exerted by a magnetic field on a conductor carrying a current is given by the integral

$$\mathbf{F} = \int \mathbf{j} \times \mathbf{H} dV / c. \quad (35.5)$$

Formula (35.4) can, of course, be very easily obtained from the familiar expression for the Lorentz force. The macroscopic force on a body at rest in a magnetic field is just the averaged Lorentz force exerted on the charged particles in the body by the microscopic field  $\mathbf{h}$ :  $\mathbf{f} = \rho \mathbf{v} \times \mathbf{h} / c$ . For  $\mu = 1$  the field  $\mathbf{h}$  is equal to the mean field  $\mathbf{H}$ , and the mean value of  $\rho \mathbf{v}$  is the conduction current density.

When a conductor moves, the forces (35.4) do mechanical work on it. At first sight it might appear that this contradicts the result that the Lorentz forces do no work on moving charges. In reality, of course, there is no contradiction, since the work done by the Lorentz forces in a moving conductor includes not only the mechanical work but also the work done by the electromotive forces induced in the conductor during its motion. These two quantities of work are equal and opposite; see the second footnote to §63.

In the expression (35.4)  $\mathbf{H}$  is the true value of the magnetic field due both to external sources and to the currents themselves on which the force (35.4) acts. In calculating the total force from (35.5), however, we can take  $\mathbf{H}$  to be simply the external field  $\mathfrak{H}$  in which the conductor carrying a current is placed. The field of the conductor itself cannot, by the law of conservation of momentum, contribute to the total force acting on the conductor.

The calculation of the forces is particularly simple for a linear conductor. Its magnetic properties are of no significance, and, if  $\mu = 1$  in the surrounding medium, the total force on the conductor is given by the line integral

$$\mathbf{F} = J \oint d\mathbf{l} \times \mathfrak{H}/c. \quad (35.6)$$

This expression can be written as an integral over a surface bounded by the current circuit. Using Stokes' theorem, we replace  $d\mathbf{l}$  by the operator  $d\mathbf{f} \times \text{grad}$ , obtaining  $\oint d\mathbf{l} \times \mathfrak{H} = \int (d\mathbf{f} \times \text{grad}) \times \mathfrak{H}$ . Now

$$\begin{aligned} (d\mathbf{f} \times \text{grad}) \times \mathfrak{H} &= -d\mathbf{f} \text{div } \mathfrak{H} + \text{grad} (d\mathbf{f} \cdot \mathfrak{H}) \\ &= -d\mathbf{f} \text{div } \mathfrak{H} + d\mathbf{f} \times \text{curl } \mathfrak{H} + (d\mathbf{f} \cdot \text{grad}) \mathfrak{H}. \end{aligned}$$

But  $\text{div } \mathfrak{H} = 0$ , and in the space outside the currents  $\text{curl } \mathfrak{H} = 0$  also. Thus

$$\mathbf{F} = J \int (d\mathbf{f} \cdot \text{grad}) \mathfrak{H}/c. \quad (35.7)$$

In particular, in an almost uniform external field  $\mathfrak{H}$  can be taken outside the integral, together with the operator  $\text{grad}$ . With the magnetic moment of the current given by (30.18), we then have the obvious result

$$\mathbf{F} = (\mathcal{M} \cdot \text{grad}) \mathfrak{H}. \quad (35.8)$$

Since  $\mathcal{M}$  in this formula is constant, we can also write

$$\mathbf{F} = \text{grad}(\mathcal{M} \cdot \mathfrak{H}), \quad (35.9)$$

in agreement with the expression (33.17) for the energy of the current. The torque acting on a current in an almost uniform field is easily seen to be given by the usual expression

$$\mathbf{K} = \mathcal{M} \times \mathfrak{H}. \quad (35.10)$$

## PROBLEM

Determine the force on a straight wire carrying a current  $J$  and parallel to an infinite circular cylinder with magnetic permeability  $\mu$ , radius  $a$  and axis at a distance  $b$  from the wire.

**SOLUTION.** On account of the relation, mentioned in the second footnote to §30, between two-dimensional problems of electrostatics and magnetostatics, the field of the current is obtained from the result in §7, Problem 3, by changing the notation. The field in the space round the cylinder is the same as that produced in a vacuum by the current  $J$  and currents  $+J'$  and  $-J'$  through  $A$  and  $O'$  (Fig. 12, §7) respectively, where  $J' = (\mu - 1)J/(\mu + 1)$ . The field within the cylinder is the same as that due to a current  $J'' = 2J/(\mu + 1)$  through  $O$ . The force per unit

length of the conductor is

$$F = JB/c = \frac{2JJ'}{c^2} \left( \frac{1}{OA} - \frac{1}{OO'} \right) \\ = \frac{2J^2 a^2 (\mu - 1)}{b(b^2 - a^2)(\mu + 1)c^2}.$$

Similarly we find (see §7, Problem 4) that a linear conductor passing through a cylindrical hole in a magnetic medium is attracted to the nearest surface of the hole by a force

$$F = 2J^2 b(\mu - 1)/(a^2 - b^2)(\mu + 1)c^2.$$

### §36. Gyromagnetic phenomena

Uniform rotation of a body (having no magnetic structure) causes a magnetization which is linearly dependent on the angular velocity  $\Omega$  (the *Barnett effect*). Phenomenologically, a linear relation between the vectors  $\mathcal{M}$  and  $\Omega$  is possible because both change sign under time reversal. Since both are axial vectors, the relation can hold even in an isotropic body, where it reduces to a simple proportionality between  $\mathcal{M}$  and  $\Omega$ .

There must also be an inverse effect: a freely suspended body, on being magnetized, begins to rotate (the *Einstein-de Haas effect*). There is a simple thermodynamic relation between the two effects; it can be derived as follows.

As we know (see SP 1, §26), the thermodynamic potential with respect to the angular velocity (for given temperature and volume of the body) is the free energy  $\tilde{\mathcal{F}}'$  of the body in a system of coordinates rotating with it. The angular momentum  $\mathbf{L}$  of the body is

$$\mathbf{L} = -\partial\tilde{\mathcal{F}}'/\partial\Omega. \quad (36.1)$$

The gyromagnetic phenomena are described by adding to the free energy a further term, which is the first term containing both  $\Omega$  and  $\mathbf{M}$  in an expansion in powers of  $\Omega$  and of the magnetization  $\mathbf{M}$  at each point in the body. This term is linear in both, i.e. it is

$$\tilde{\mathcal{F}}'_{\text{gyro}} = -\int \lambda_{ik} \Omega_i M_k dV = -\lambda_{ik} \Omega_i \mathcal{M}_k, \quad (36.2)$$

where  $\lambda_{ik}$  is a constant tensor, in general unsymmetrical.

According to (36.1) and (36.2) the angular momentum acquired by the body as a result of magnetization is related to its total magnetic moment by  $L_{\text{gyro},i} = \lambda_{ik} \mathcal{M}_k$ . It is usual to replace  $\lambda_{ik}$  by the inverse tensor, defined as  $g_{ik} = (2mc/e)\lambda^{-1}_{ik}$ , where  $e$  and  $m$  are the electron charge and mass. The dimensionless quantities  $g_{ik}$  are called *gyromagnetic coefficients*. Then

$$\mathcal{M}_i = (e/2mc)g_{ik} L_{\text{gyro},k}. \quad (36.3)$$

The expression (36.2) also shows that, as regards its magnetic effect, the rotation of the body is equivalent to an external field  $\mathfrak{H}_i = \lambda_{ki} \Omega_k$  or

$$\mathfrak{H}_i = (2mc/e)g^{-1}_{ki} \Omega_k. \quad (36.4)$$

We thus have the possibility, in principle, of calculating the magnetization caused by the rotation. For example, if the magnetic susceptibility  $\chi_{ik}$  of the body is small, the magnetic moment which it acquires is independent of its shape and is

$$\mathcal{M}_i = \chi_{ik} \mathfrak{H}_k = (2mc/e)\chi_{ik} g^{-1}_{lk} \Omega_l.$$

Formulae (36.3) and (36.4) represent respectively the Einstein-de Haas and Barnett effects. We see that both effects are determined by the same tensor  $g_{ik}$ .

## CHAPTER V

# FERROMAGNETISM AND ANTIFERROMAGNETISM

### §37. Magnetic symmetry of crystals

THERE is a profound difference between the electric properties of crystals and their magnetic properties, which results from a difference in the behaviour of charges and currents with respect to time reversal.

The invariance of the equations of motion with respect to this change means that the formal substitution  $t \rightarrow -t$ , on being applied to any state of thermodynamic equilibrium of a body, must give some possible equilibrium state. There are then two possibilities: either the state obtained by changing the sign of  $t$  is the same as the original state, or it is not.

In this section we denote by  $\rho(x, y, z)$  and  $\mathbf{j}(x, y, z)$  the true (microscopic) charge and current densities at any given point in the crystal, averaged only over time, and not over "physically infinitesimal" volumes as in the macroscopic theory. These are the functions which determine the electric and magnetic structure of the crystal respectively.

When  $t$  is replaced by  $-t$ ,  $\mathbf{j}$  changes sign. If the state of the body remains unchanged, it follows that  $\mathbf{j} = -\mathbf{j}$ , i.e.  $\mathbf{j} = 0$ . Thus there is a reason why bodies can exist in which the function  $\mathbf{j}(x, y, z)$  is identically zero. In such bodies, not only the current density but also the (time) average magnetic field and magnetic moment vanish at every point (we are speaking, of course, of states in the absence of an external magnetic field). Such bodies may be said to have no "magnetic structure", and indeed the great majority of bodies fall into this category.

The charge density  $\rho$ , on the other hand, is unchanged when  $t \rightarrow -t$ . There is therefore no reason why this function should be identically zero. In other words, there are no crystals without "electric structure", and herein lies the essential difference, mentioned at the beginning of this section, between the electric and the magnetic properties of crystals.

Let us now consider crystals for which the change from  $t$  to  $-t$  results in a change of state, so that  $\mathbf{j} \neq 0$ . We shall say that such bodies have a magnetic structure. First of all, we note that, although  $\mathbf{j}$  is not zero, there can be no total current in an equilibrium state of the body, i.e. the integral  $\int \mathbf{j} dV$  taken over a unit cell must always be zero.† Otherwise the current would produce a macroscopic magnetic field, and the crystal would have a magnetic energy per unit volume increasing rapidly with its dimensions. Since such a state is energetically unfavourable, it could not correspond to thermodynamic equilibrium.

The currents  $\mathbf{j}$  may, however, produce a non-zero macroscopic magnetic moment, i.e.

† It should be emphasized that the cell spoken of here is the true unit cell, whose definition involves the magnetic structure of the crystal, and this "magnetic cell" may be different from the purely crystallographic cell, which relates only to the symmetry of the charge distribution in the lattice (cf. §38 below).

the integral  $\int \mathbf{r} \times \mathbf{j} dV$ , again taken over a unit cell, need not be zero. Accordingly, the bodies for which  $\mathbf{j} \neq 0$  may be divided into two types: those in which the macroscopic magnetic moment is not zero, called *ferromagnets*, and those in which it is zero, called *antiferromagnets*.

What are the possible symmetry groups of the current distribution  $\mathbf{j}(x, y, z)$ ? This symmetry contains, first of all, the usual rotations, reflections and translations, and so the possible symmetry groups of  $\mathbf{j}$  always include the usual 230 crystallographic space symmetry groups. These, however, are by no means all. As has already been mentioned, the substitution  $t \rightarrow -t$  changes the sign of the vector  $\mathbf{j}$ . For this reason a new symmetry element comes in, namely that resulting from the reversal of all currents; we shall denote this transformation by  $R$ . If the current distribution itself has the symmetry  $R$ , it follows that  $\mathbf{j} = -\mathbf{j}$ , i.e.  $\mathbf{j} = 0$ , and the body has no magnetic structure. A non-zero function  $\mathbf{j}(x, y, z)$  may, however, be symmetrical with respect to various combinations of  $R$  with the other symmetry elements (rotations, reflections and translations). Thus the problem of determining the possible types of symmetry of the current distribution (the *magnetic space groups*) amounts to the enumeration of all possible groups containing both the transformations of the ordinary space groups and the combinations of these with  $R$ .

If the symmetry of the current distribution is given, the crystallographic symmetry of the particle distribution, which is also the symmetry of the function  $\rho(x, y, z)$ , is determined. It is the symmetry of the space group which is obtained from the symmetry group of  $\mathbf{j}$  by formally regarding the transformation  $R$  as the identity (as it is with respect to the function  $\rho$ ).

If only the macroscopic properties of the body are of interest, however, it is not necessary to know the complete symmetry group of the function  $\mathbf{j}(x, y, z)$ . These properties depend only on the direction in the crystal, and the translational symmetry of the lattice does not affect them. As regards crystallographic structure, the "symmetry of directions" is specified by the 32 crystal classes. These are the symmetry groups consisting of rotations and reflections only, and are obtained from the space groups by regarding every translation as the identity, and the screw axes and glide planes as simple axes and planes of symmetry. As regards the magnetic properties, the macroscopic symmetry can be classified by groups (consisting of rotations, reflections and combinations of these with  $R$ ) which may be called the *magnetic crystal classes*. They are related to the magnetic space groups in the same way as are the ordinary crystal classes to the ordinary space groups. They include, firstly, the usual 32 classes, and those classes augmented by the element  $R$ . These augmented classes are, in particular, the macroscopic symmetry groups for all bodies having no magnetic structure, but they occur also in bodies with magnetic structure. This happens if the magnetic space symmetry group of such bodies includes  $R$  only in combination with translations, and not alone.

There are also 58 classes in which  $R$  enters only in combination with rotations or reflections. Each of these becomes one of the ordinary crystal classes if  $R$  is replaced by the identity.

It should be noted that the occurrence of magnetic structure (ferromagnetic or antiferromagnetic) always involves comparatively weak interactions.† Hence the crystal

† The exchange interaction between the magnetic moments of atoms usually results in the saturation of the valency bonds and the formation of non-magnetic structures. A magnetic structure results only from the relatively weak exchange interactions between deep-lying  $d$  and  $f$  electrons of atoms of elements in the intermediate groups of the periodic system.

structure of a magnetic body is only a slight modification of that in the non-magnetic phase, which usually changes into the magnetic phase when the temperature is reduced. In this respect ferromagnets, in particular, differ from ordinary pyroelectrics, but are analogous to ferroelectrics.

If the magnetic crystal class of a body is specified, its macroscopic magnetic properties are qualitatively determined. The most important of these is the presence or absence of a macroscopic magnetic moment, i.e. of spontaneous magnetization in the absence of an external field. The magnetic moment  $\mathbf{M}$  is a vector, behaving as an axial vector (the vector product of two polar vectors) under rotation and reflection, and changing sign under the operation  $R$ . The crystal will possess spontaneous magnetization if it has one or more directions such that a vector  $\mathbf{M}$  in that direction and having the above-mentioned properties is invariant under all transformations belonging to the magnetic crystal class concerned.

We must again emphasize the difference between these (macroscopic) properties and the corresponding ones in electrostatics. The latter are qualitatively determined by the ordinary crystal class. In particular, a body is pyroelectric if its crystal class admits the existence of a polar vector  $\mathbf{P}$  (the polarization). It would, however, be entirely wrong to base conclusions about the existence or otherwise of a macroscopic magnetic moment on the behaviour of the axial vector  $\mathbf{M}$  with respect to the transformations of the purely structural crystal class of the body concerned, which corresponds to the symmetry of the function  $\rho(x, y, z)$ ; we shall return to this problem in §38, after actually constructing the magnetic classes.

Instead of the symmetry of the function  $\mathbf{j}(x, y, z)$ , we can use that of the microscopic magnetic moment density distribution  $\mathbf{M}(x, y, z) = \mathbf{r} \times \mathbf{j}(x, y, z)$ . This in turn may be regarded as the symmetry of the configuration and orientation of the (time) averaged values of the magnetic moments  $\mu$  of the atoms or ions in the lattice. In a body without magnetic symmetry these averaged values are zero. In a ferromagnet the sum of the atomic moments in each unit cell is non-zero, but in an antiferromagnet it is zero.

A set of atoms in the lattice having equal values of  $\mu$  is called a *magnetic sub-lattice*. An antiferromagnet will evidently contain at least two sub-lattices with equal and opposite values of  $\mu$ . If the directions of the moments  $\mu$  are parallel or antiparallel for all sub-lattices, such a body is said to be *collinear*; in the contrary case, it is a *non-collinear* antiferromagnet.

A ferromagnet also may contain more than one sub-lattice. In the narrow sense, the term "ferromagnetic" applies to bodies in which all the mean atomic magnetic moments are parallel. A crystal is said to be *ferrimagnetic* if it contains two or more sub-lattices with atomic moments that differ in direction or magnitude. In contrast to the antiferromagnetic case, in these substances the vector sum  $\mathbf{M}$  of the magnetic moments of the sub-lattices is not zero. A ferromagnet may be collinear, or not, according as the magnetic moments of all its sub-lattices are parallel or antiparallel, or not.

### §38. Magnetic classes and space groups

We shall now show how the magnetic symmetry groups are constructed, beginning with the magnetic classes.

As already mentioned in §37, the magnetic classes fall into three types. Type I includes the 32 ordinary crystal classes, which do not involve the element  $R$ . Type II includes the same 32 classes augmented by  $R$ . Each such class contains all the elements of the ordinary



class (of the point group  $G$ ) and also all these elements multiplied by  $R$ ; if the magnetic class is denoted by  $M$ , then we can write

$$M = G + RG. \quad (38.1)$$

The transformation  $R$  commutes, of course, with all spatial rotations and reflections; hence  $RG = GR$ , where  $G$  is any element of  $G$ .

These two types of class are, in a sense, trivial. The non-trivial type III includes the 58 magnetic classes, in which  $R$  enters only in combination with rotations or reflections. Each of these becomes one of the ordinary crystal classes of  $G$  if  $R$  is replaced by the identity. The construction of all magnetic classes of this type is carried out as follows.

Let  $H$  denote the set of elements of the group  $G$  which are not multiplied by  $R$  when the magnetic class  $M$  is formed. Such a set includes, by definition, the unit element  $E$ , since otherwise  $M$  would contain the element  $R$  itself, i.e., would belong to type II. The products of any two of the elements in this set are also members of the set. Thus  $H$  is a sub-group of  $G$ . All other elements of  $G$  appear in  $M$  multiplied by  $R$ ; since  $R^2 = E$ , all products of pairs of these elements belong to  $H$ . It follows that  $H$  is a sub-group of  $G$  (and therefore of  $M$ ) with index 2.† The structure of a type III magnetic class  $M$  may therefore be written as

$$M = H + RG_1H, \quad (38.2)$$

where  $G_1$  is any element of  $G$  that is not an element of  $H$ . It is evident that the groups  $M$  and  $G = H + G_1H$  are isomorphous.

The problem of constructing all the magnetic classes thus amounts to finding the sub-groups with index 2 in all the crystal classes. This is easily done by means of the character tables for the irreducible representations of the point groups. Every one-dimensional (other than the unit) representation of the group contains an equal number of characters  $+1$  and  $-1$ ; the elements with characters  $+1$  form a sub-group with index 2. On changing to the magnetic class, these elements remain the same, while all others are multiplied by  $R$ .

The procedure may be illustrated with reference to the point group  $C_{4v}$ . The character table for its irreducible representations is (see *QM*, §95)

|       | $E$ | $C_2$ | $2C_4$ | $2\sigma_v$ | $2\sigma'_v$ |
|-------|-----|-------|--------|-------------|--------------|
| $A_1$ | 1   | 1     | 1      | 1           | 1            |
| $A_2$ | 1   | 1     | 1      | -1          | -1           |
| $B_1$ | 1   | 1     | -1     | 1           | -1           |
| $B_2$ | 1   | 1     | -1     | -1          | 1            |
| $E$   | 2   | -2    | 0      | 0           | 0            |

The one-dimensional (other than the unit) representations are  $A_2$ ,  $B_1$ ,  $B_2$ . In the representation  $A_2$ , the elements with characters  $+1$  form the sub-group  $C_4$ . The corresponding magnetic class, denoted by  $C_{4v}(C_4)$ , consists of the elements  $E$ ,  $C_2$ ,  $2C_4$ ,  $2R\sigma_v$ ,  $2R\sigma'_v$ . In the representations  $B_1$  and  $B_2$ , the elements with characters  $+1$  form the sub-groups  $C_{2v}$ , which differ only in the position of the planes  $\sigma_v$  relative to fixed coordinates. These sub-groups are crystallographically indistinguishable, and correspond to the same magnetic class  $C_{4v}(C_{2v})$ , whose elements are  $E$ ,  $C_2$ ,  $2RC_4$ ,  $2\sigma_v$ ,  $2R\sigma'_v$ .

† This means that  $H$  contains half as many elements as  $G$ . The statement is a consequence of a general theorem which is fairly obvious: a sub-group  $H$  of  $G$  has index 2 if and only if the product of any two elements of  $G$  that do not belong to  $H$  is an element of  $H$ .

TABLE 1 Magnetic classes

|   |   |
|---|---|
| $C_i(C_1)$                                    | $C_{3v}(C_3)$                                 |
| $C_s(C_1)$                                    | $D_3(C_3)$                                    |
| $C_2(C_1)$                                    | $D_{3d}(D_3, S_6, C_{3v})$                    |
| $C_{2h}(C_i, C_2, C_s)$                       | $C_{3h}(C_3)$                                 |
| $C_{2v}(C_s, C_2)$                            | $C_6(C_3)$                                    |
| $D_2(C_2)$                                    | $D_{3h}(C_{3h}, C_{3v}, D_3)$                 |
| $D_{2h}(D_2, C_{2h}, C_{2v})$                 | $C_{6h}(C_6, S_6, C_{3h})$                    |
| $C_4(C_2)$                                    | $C_{6v}(C_6, C_{3v})$                         |
| $S_4(C_2)$                                    | $D_6(C_6, D_3)$                               |
| $D_{2d}(S_4, D_2, C_{2v})$                    | $D_{6h}(D_6, C_{6h}, C_{6v}, D_{3d}, D_{3h})$ |
| $D_4(C_4, D_2)$                               | $T_h(T)$                                      |
| $C_{4v}(C_4, C_{2v})$                         | $O(T)$  |
| $C_{4h}(C_4, C_{2h}, S_4)$                    | $T_d(T)$                                      |
| $D_{4h}(D_4, C_{4h}, D_{2h}, C_{4v}, D_{2d})$ | $O_h(O, T_h, T_d)$                            |
| $S_6(C_3)$                                    |   |

On going through all the 32 crystal classes in this way, we obtain the 58 magnetic classes of type III, as listed in Table 1. Each class  $G(H)$  is defined by an original point group  $G$  and a sub-group  $H$  thereof which is one of those listed in parentheses after the symbol for the particular group  $G$ . The crystal classes  $C_1$ ,  $C_3$  and  $T$  have no sub-groups with index 2, and so they give rise to no magnetic classes. The rotation  $C_3$ , moreover, never appears multiplied by  $R$  in a (non-trivial) magnetic class: the rotation  $C_3 R$  thrice repeated would give  $R$ , which is not a member of such classes.†

It has been mentioned in §37 that the crystal class concerned does not determine whether or not ferromagnetism can exist. To illustrate this, let us consider a tetragonal lattice of identical atoms, with magnetic moments parallel to the tetragonal axis.‡ Its magnetic crystal class is  $D_{4h}(D_4)$ , containing the transformations§  $E, C_2, 2C_4, 2U_2 R, 2U'_2 R, I, \sigma_h, 2S_4, 2\sigma_v R, 2\sigma'_v R$ . All these transformations leave invariant the axial vector  $M$ , which is parallel to the fourfold axis. The crystal class  $D_{4h}$  itself, however, would not allow the existence of an axial vector: all the components  $M_x, M_y, M_z$  would change sign in a rotation about some twofold axis, for example.

Let us now turn to the magnetic space groups. These are in the same relation to the ordinary crystal space groups as the magnetic classes are to the crystal classes, reducing to these if  $R$  is replaced by the identity. The total number of magnetic space groups is 1651: like the magnetic classes, they fall into three types.

Type I contains 230 groups which coincide with the crystal groups and do not involve  $R$ ; type II has the same 230 groups augmented by  $R$ .

The non-trivial type III contains 1191 groups in which  $R$  enters only in combination with rotations, reflections or translations. It has the structure (38.2), where  $H$  is any sub-group

† In abstract symmetry theory, magnetic symmetry is called antisymmetry. This concept was independently proposed by H. Heesch (1929) and A. V. Shubnikov (1945). The antisymmetry classes were found by Shubnikov (1951) as the symmetry groups of geometric figures (polyhedra) with faces of two colours; the element  $R$  then corresponds to changing the colour of each face. These classes were derived as magnetic symmetry groups by B. A. Tavger and V. M. Zaitsev (1956). The method of derivation given here is due to V. L. Indenbom (1959).

‡ Such, for instance, is the lattice of iron in its ferromagnetic phase. Crystallographically, it is a cubic lattice slightly distorted along one of the fourfold symmetry axes. The distortion is the result of magnetostriction arising simply from the presence of the magnetic structure.

§ The notation for the symmetry elements is everywhere as in *QM*, §§93 and 94. In particular,  $U_2$  and  $U'_2$  are rotations through  $180^\circ$  about horizontal axes perpendicular to the fourfold axis,  $\sigma_h$  is a reflection in the horizontal plane,  $\sigma_v$  and  $\sigma'_v$  are reflections in vertical planes passing through the axes  $C_4$  and  $U_2, U'_2$ . The symbols  $U_2 R, \sigma_v R$  etc. denote symmetry planes and axes which appear in this class in combination with  $R$ .



with index 2 of the crystal space group  $G$ , and  $G_1$  is an element of  $G$  that is not an element of  $H$ . The sub-group  $H$  must evidently coincide with the original space group  $G$  as regards either translational symmetry or class, having respectively either half as many "rotational" elements (rotations and reflections) as  $G$ , or half as many translations. Accordingly, type III may be subdivided into two sub-types.

Sub-type IIIa contains the magnetic space groups for which  $G_1$  in (38.2) is a "rotational" transformation of the crystal group  $G = H + G_1 H$  which does not belong to  $H$ . The translational symmetry (Bravais lattice) of a space group  $M$  of this type is the same as that of  $G$ ; that is, the unit cell of the magnetic structure is the same as the purely crystallographic one. These magnetic space groups, 674 in number, belong to magnetic classes of type III.

Sub-type IIIb contains the magnetic space groups for which  $G_1$  in (38.2) may be taken to be a pure translation through one of the basic periods of  $G$ . The unit cell of the magnetic structure has a volume twice that of the crystallographic unit cell. The set of pure translations and translations multiplied by  $R$  forms the *magnetic Bravais lattice*; there are 22 different lattices of this kind. The magnetic space groups of sub-type IIIb, 517 in number, belong to magnetic classes of type II.†

## PROBLEM

List the magnetic classes that allow ferromagnetism.

**SOLUTION.** The classes of type II do not allow ferromagnetism. It must be emphasized that the same is therefore true of every space group of type II or IIIb that contains translations multiplied by  $R$ ; these transformations certainly change the sign of  $M$ . In other words, for ferromagnetism to occur, it is always necessary that the magnetic unit cell should coincide with the crystallographic one.‡

Of the classes of type I (which are the same as the ordinary crystal classes), the following allow the existence of an axial vector  $M$ :  $C_1$  and  $C_i$  with  $M$  in any direction;  $C_s$  with  $M$  in the plane of symmetry;  $C_2$ ,  $C_{2h}$ ,  $C_4$ ,  $S_4$ ,  $C_{4h}$ ,  $C_3$ ,  $C_6$ ,  $S_6$ ,  $C_{3h}$ ,  $C_{6h}$  with  $M$  parallel to the axis of symmetry.

For ferromagnetism to exist in classes of type III, they must never contain the element  $IR$ , which changes the sign of the vector  $M$  whatever the direction of the latter. Of such classes, the following allow ferromagnetism:  $C_2(C_1)$ ,  $C_{2h}(C_i)$  with  $M$  perpendicular to the  $C_2 R$  axis;  $C_s(C_1)$  with  $M$  in the  $\sigma R$  plane;  $C_{2v}(C_s)$  with  $M$  in the  $\sigma_v R$  plane and perpendicular to the  $C_2 R$  axis;  $D_2(C_2)$ ,  $C_{2v}(C_2)$ ,  $D_{2h}(C_{2h})$  with  $M$  parallel to the  $C_2$  axis;  $D_4(C_4)$ ,  $C_{4v}(C_4)$ ,  $D_{2d}(S_4)$ ,  $D_{4h}(C_{4h})$ ,  $D_3(C_3)$ ,  $C_{3v}(C_3)$ ,  $D_{3d}(S_6)$ ,  $D_{3h}(C_{3h})$ ,  $D_6(C_6)$ ,  $C_{6v}(C_6)$ ,  $D_{6h}(C_{6h})$  with  $M$  parallel to the  $C_4$ ,  $C_3$  or  $C_6$  axis.

## §39. Ferromagnets near the Curie point

There is a close analogy between the magnetic properties of ferromagnets and the electric properties of ferroelectrics. Both exhibit spontaneous polarization, magnetic or electric, in macroscopic volumes. In each case, this polarization vanishes at a temperature corresponding to a second-order phase transition (the transition point between the ferromagnetic and paramagnetic phases is called the *Curie point*).

There are also, however, important differences between ferromagnetic and ferroelectric phenomena, arising from the difference in the microscopic interaction forces which bring

† The magnetic space groups were constructed (as antisymmetry groups) by A. M. Zamorzaev (1953) and by N. V. Belov, N. N. Neronova, and T. S. Smirnova (1955); the latter authors' tables are given by A. V. Shubnikov and N. V. Belov, *Colored Symmetry*, Pergamon Press, Oxford, 1964. The most complete tables of the magnetic space groups and their properties are those given by V. A. Koptsik, *Shubnikov Groups (Shubnikovskie gruppy)*, Moscow State University Press, 1966.

‡ This refers (see the last footnote but two) to the symmetry of a lattice already distorted by the very existence of the magnetic structure.

about the spontaneous polarization. In ferroelectrics, the interaction between the molecules in the crystal lattice is essentially anisotropic, and consequently the spontaneous polarization vector is fairly closely related to certain directions in the crystal. The formation of a magnetic (including ferromagnetic) structure, on the other hand, is due mainly to the exchange interaction of the atoms, which is quite independent of the direction of the total magnetic moment relative to the lattice.† It is true that, together with the exchange interaction, there is also a direct magnetic interaction between the magnetic moments of the atoms. This interaction, however, is an effect of order  $v^2/c^2$  ( $v$  being the atomic velocities), since the magnetic moments themselves contain a factor  $v/c$ . The effects of this order include also the interaction of the magnetic moments of the atoms with the electric field of the crystal lattice. All these interactions, which may be called *relativistic* by virtue of the factor  $1/c^2$  in them, are weak in comparison with the exchange interaction, so that they can result only in a comparatively slight dependence of the energy of the crystal on the direction of magnetization. This relationship between the exchange and relativistic interactions will be assumed throughout the rest of the chapter.‡

Consequently, the magnetization of a ferromagnet is a quantity which, in the first approximation (i.e. on the basis of the exchange interaction), is conserved. This fact endows with greater physical significance the thermodynamic theory, in which the magnetization  $\mathbf{M}$  is regarded as an independent variable, the actual value of which (as a function of temperature, field, etc.) is afterwards determined by the appropriate conditions of thermal equilibrium.§

We denote by  $\Phi(\mathbf{M}, \mathbf{H})$  the thermodynamic potential per unit volume of the substance, regarded as a function of the independent variable  $\mathbf{M}$  (and of the other thermodynamic variables). We shall, for the present, neglect the relativistic interactions, i.e. we shall take into account only the exchange interaction. Then  $\Phi(\mathbf{M}, 0)$  may be a function of the magnitude of  $\mathbf{M}$ , but not of its direction.

In order to find the thermodynamic quantities when the field  $\mathbf{H}$  is not zero, we proceed exactly as in the derivation of (19.3), starting from the relation  $\partial\tilde{\Phi}/\partial\mathbf{H} = -\mathbf{B}/4\pi$ . This gives

$$\tilde{\Phi}(\mathbf{M}, \mathbf{H}) = \Phi(M, 0) - \mathbf{M} \cdot \mathbf{H} - H^2/8\pi. \quad (39.1)$$

Hence the potential  $\Phi$  is

$$\begin{aligned} \Phi(\mathbf{M}, \mathbf{B}) &= \tilde{\Phi} + \mathbf{H} \cdot \mathbf{B}/4\pi \\ &= \Phi(M, 0) + H^2/8\pi \\ &= \Phi(M, 0) + (\mathbf{B} - 4\pi\mathbf{M})^2/8\pi. \end{aligned} \quad (39.2)$$

† The exchange interaction is a quantum effect resulting from the symmetry of the wave functions of the system with respect to interchanges of the particles. The interchange symmetry of the wave functions, and therefore the exchange interaction, depend only on the total spin of the system, and not on the direction of the spin; see *QM*, §60. The importance of the exchange interaction in ferromagnets was first pointed out by Ya. I. Frenkel, Ya. G. Dorfman and W. Heisenberg (1928).

‡ The order of magnitude of the ratio of the relativistic and exchange interactions is given by the ratio  $U_{\text{aniso}}/NT_c$ , where  $U_{\text{aniso}}$  is the magnetic anisotropy energy (see §40),  $N$  the number of atoms per unit volume, and  $T_c$  the temperature of the Curie point. For ferromagnets this ratio is usually between  $10^{-2}$  and  $10^{-5}$ . In some ferromagnets (rare-earth metals and their compounds), however, it may be much greater and even reach values of the order of unity, both because of the "anomalously" large anisotropy energy and because of the relative weakness of the exchange interactions. There are, of course, only limited possibilities of applying the macroscopic theory to such substances. A detailed discussion of the microscopic mechanisms of the various interactions in specific magnetic materials is outside the scope of this book.

§ The collinear ferrimagnets are indistinguishable from ferromagnets in the narrow sense (see the end of §37) as regards macroscopic magnetic symmetry and as regards their behaviour in fairly weak magnetic fields. The theory given below relates to both these types of material.